Pathwise Stochastic Integrals for Model Free Finance

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Usual approach to pricing an option with maturity T and underlying $(S_t)_{t\in[0,T]}$ and pay-off $\Phi(T)$ at maturity T:

1. Select a class of models $\mathfrak{M}$ suitable for the problem. Each $M(\theta) \in \mathfrak{M}$ is specified by a finite dimensional parameter vector θ.
2. Calibrate the model using market data i.e. choose $\hat{\theta}$ s.t. the mean-square error between $M(\hat{\theta})$ and the market prices of the traded options is minimal.
3. Compute the option value as $p_\Phi = \mathbb{E}_{\hat{\theta}}[e^{-rT}\Phi(T)]$ (under the model $M(\hat{\theta})$).

Problem: Different models may give different prices!

$\Rightarrow$ What can we say about the value of an option with pay-off given only current market data?

Challenges of Model Free Finance?

$\Rightarrow$ get rid of probabilistic model, **but** (at least) two problems:

- What does "no arbitrage" even mean if there is nothing like risk?
- how to interpret SDEs like

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = \xi \tag{1}$$

without the Ito framework?

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be s.t. S is a semi-martingale. Then the following are equivalent

- $\exists$ *equivalent local-martingale measure* $\mathbb{Q}$
- S *satisfies no free lunch with vanishing risk (NFLVR)*
- *no arbitrage (NA) & no arbitrage of the first kind (NA1)*

"The crucial point is that (NA1) is the essential property which every market model has to satisfy, whereas (NA) is nice to have but not strictly necessary" (p. 9, [PP16]).

$\Rightarrow$ Reformulate arbitrage as what can be replicated without infinite debt, along the lines of (NA1)

Integration Problem

More generally, when does

$$\int_0^t H_s dM_s = \lim_{|P|\downarrow 0} \sum_{s,t\in P} [H_s(M_t - M_s)] \tag{2}$$

make sense?

Framework	H	M
Basic	simple	anything
Riemann	C^0	C^1
Riemann-Stieltjes	C^0	BV
Young $\alpha + \beta > 1$	C^α	C^β
Föllmer	$\nabla F(M.)$	quad. variation $< \infty$
Ito	cont. adapted	cont. L^2 semi-martingale

Integration Problem: Solution?

Two proposed solutions:

(1) Topology on space of paths making the integral continuous

+ no restriction of possible price paths

- not a Banach space theory

(2) Rough path integral

+ Banach space theory, extending Riemann Stieltjes, Young, Föllmer, ...

- restricted set of price paths, but still defined for all "typical" price paths

- no immediate financial interpretation of compensation terms

- needs an a priori definition of integration for second-order increment

Notation I

- $T \in (0, \infty)$
- $\omega \in \Omega = C([0, T], \mathbb{R}^d)$
- $(S_t)_{t \in [0,T]}$ is the coordinate process i.e. $\forall \omega \in \Omega : S_t(\omega) = \omega(t)$
- $(\mathcal{F}_t)_{t \in [0,T]}$ is the natural filtration with $\mathcal{F}_t := \sigma(S_s : s \leq t)$

Notation II

Recall: A process $H : \Omega \times [0,T] \to \mathbb{R}^d$ is called **simple strategy** if $\exists 0 = \tau_0 < \tau_1 < \dots$ and $\mathcal{F}_t$-measurable bounded functions $F_n : \Omega \to \mathbb{R}^d$ s.t. for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many $n \in \mathbb{N}$ and

$$H_t(\omega) := \sum_{n\in\mathbb{N}} F_n(\omega) 1_{(\tau_n(\omega),\tau_{n+1}(\omega)]}(t), \quad (\omega,t) \in \Omega \times [0,T]. \tag{3}$$

Its integral against S is given by

$$(H \cdot S)_t(\omega) := \sum_{n\in\mathbb{N}} F_n(\omega)\big(S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)\big). \tag{4}$$

For $\lambda > 0$, a simple strategy is called λ-**admissible** ($\in \mathcal{H}_\lambda$) if $(H \cdot S)_t(\omega) \geq -\lambda$ for every $(\omega,t) \in \Omega \times [0,T]$.

Outer Measure, Superhedging, and Arbitrage

Outer Measure (not probability) ⇒ Arbitrage & Topology

Definition
Let $A \subseteq \Omega$. Then

$$\overline{P}(A) := \inf\left\{\lambda > 0 : \exists (H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda s.t. \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_T(\omega)) \geq 1_A(\omega) \forall \omega \in \Omega\right\} \tag{5}$$

is the cheapest superhedging price for 1_A.

- Note $\overline{P}(\Omega) \leq 1$ by choosing $H^n = 0$.

- $\overline{P}$ defines an outer measure on Ω.

This has nothing to do with "how likely is this path to happen"!

Typical Price Paths and Sets with $\overline{P}(A) = 0$

Sets of outer measure 0 have an arbitrage interpretation!

Lemma

Let $A \subseteq \Omega$. Then $\overline{P}(A) = 0$ if and only if there exists a sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_1$ s.t.

$$\liminf_{n\to\infty}(1 + (H^n \cdot S)_T(\omega)) \geq \infty \cdot 1_A(\omega) \tag{6}$$

i.e. $\overline{P}(A) = 0$ if infinite profit can be made from investing in the paths of A, while risking no more than 1.

"Property P holds for **typical** price paths" :$\Leftrightarrow$ "property P holds for all $\omega \notin A$"

Proof of Lemma

Proof.

Link to Classical 0-Sets

Proposition

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ s.t. S is a $\mathbb{P}$-local martingale, and let $A \in \mathcal{F}$. Then $\mathbb{P}(A) \leq \overline{P}(A)$. In particular, $\overline{P}(A) = 0$ implies $\mathbb{P}(A) = 0$.

Proof.

Model Free Arbitrage

Definition
A map $X : \Omega \to [0, \infty)$ is called **model free arbitrage opportunity** if $X \not\equiv 0$ and if there exists a $c > 0$ and a sequence $(H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_c$ s.t.

$$\liminf_{n \to \infty} (H^n \cdot S)_T(\omega) = X(\omega), \quad \omega \in \Omega. \tag{7}$$

Relation to classical arbitrage: See prop. 2.6 and 2.7

Topology and Ito Type Integration as a Continuous Operator

Classical Ito Isometry

Classically, we have the Ito isometry:

$$\mathbb{E}\left[\left(\int_0^T H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^T H_s^2 d\langle M\rangle_s\right] \leq \mathbb{E}\left[\|H^2\|_{\infty,[0,t]}\langle M\rangle_T\right] \tag{8}$$

$\Rightarrow \left\|\int_0^T H_s dM_s\right\|_2^2$ is controlled by the quadratic variation of M and the size of H.

Lemma

There exists an increasing sequence of (random) partitions (each consisting of stopping times) $(\sigma_k^n(\omega) : k \in \mathbb{N})_{n \in \mathbb{N}}$ *s.t. a typical price path* $\omega \in \Omega$ *has quadratic variation along i.e. the sequence*

$$V_t^{n,i}(\omega) := \sum_{k=0}^{\infty} \left(S_{\sigma_{k+1}^{n,i}(\omega) \wedge t}(\omega) - S_{\sigma_k^{n,i}(\omega) \wedge t}(\omega) \right)^2, \quad t \in [0,T], n \in \mathbb{N}, \tag{9}$$

converges uniformly to a function $\langle S^i \rangle \in C([0,T], \mathbb{R}^d)$ *for* $i = 1, \dots, d$ *when* $n \to \infty$.

Model Free Version of Ito's "Isometry"

Lemma

Let F be a step function and $d = \dim \mathbb{R}^d$. Then for any $a, b, c > 0$ we have

$$\overline{P}\Big(\{\|F \cdot S\|_\infty \geq ab\sqrt{c}\} \cap \{\|F\|_\infty \leq a\} \cap \{\langle S \rangle_T \leq c\}\Big) \leq 2\exp\left(-\frac{b^2}{2d}\right) \tag{10}$$

$\Rightarrow$ $\|F \cdot S\|_\infty$ can be controlled in terms of $\|F\|_\infty$ and $\langle S \rangle_T$.

Topology on Space of Processes

Define

- the set
  $\overline{L}_0([0,T],\mathbb{R}^d) := \{\Omega \times [0,T] \to \mathbb{R}^d\}/\{X_t = Y_t$ for typical price paths for every $t\}$.
- an expectation operator

$$\overline{E}[X] := \inf\{\lambda > 0 : \exists (H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda s.t. \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_T(\omega)) \geq X(\omega) \forall \omega \in \Omega\} \tag{11}$$

- a metric $d_\infty(X,Y) := \overline{E}[\|X-Y\|_\infty \wedge 1]$ making $\overline{L}_0([0,T],\mathbb{R}^d)$ into a complete metric space

Integral as Continuous Operator?

Want $H \mapsto (H \cdot S)$ to be continuous w.r.t.

$$(\overline{L}_0([0,T],\mathbb{R}^d), d_\infty) \to (\overline{L}_0([0,T],\mathbb{R}^d), d_\infty). \tag{12}$$

but that does not work!

Example

Intuition? Solution?

Consider the situation for ds on $[0, \infty)$ instead of dS on Ω.

We have

$$\left\| \frac{1}{n} \right\|_\infty \to 0 \tag{13}$$

But

$$\left\| \int_0^t \frac{1}{n} ds \right\|_\infty = \frac{1}{n} \cdot t \tag{14}$$

does **not** converge uniformly over t.

Solution: Consider compact convergence i.e. $\|f_n 1_K\|_\infty \to 0 \;\; \forall$ compact $K \subseteq [0, \infty)$

Integral as Continuous Operator?

Instead, use control of $H \mapsto (H \cdot S)$ via $\langle S \rangle_T$!

$$(\overline{L}_0([0,T], \mathbb{R}^d), d_\infty) \to (\overline{L}_0([0,T], \mathbb{R}^d), d_{loc}). \tag{15}$$

with

$$d_{loc}(X,Y) := \sum_{i=0}^{\infty} 2^{-n} \underbrace{\overline{E}[(\|X - Y\|_\infty \wedge 1) 1_{\{\langle S \rangle_T \leq 2^{-n}\}}]}_{=: d_{2^{-n}}(X,Y)} \tag{16}$$

Integral as Continuous Operator!

Idea: Let $F, G \in \overline{L}_0$ be simple. For any $c > 0$

$$
\begin{aligned}
d_c((F \cdot S), (G \cdot S)) &= \overline{E}[\|(F-G)\cdot S\|_\infty 1_{\{\langle S\rangle_T \le c\}}] \\
&\le \overline{E}[\|(F-G)\cdot S\|_\infty 1_{\{\langle S\rangle_T \le c\}} 1_{\{\|F-G\|_\infty \le a\}} 1_{\{\|(F-G)\cdot S\|_\infty \ge ab\sqrt{c}\}}] \\
&+ \overline{E}[\|(F-G)\cdot S\|_\infty 1_{\{\langle S\rangle_T \le c\}} 1_{\{\|F-G\|_\infty \ge a\}} 1_{\{\|(F-G)\cdot S\|_\infty \ge ab\sqrt{c}\}}] \\
&+ \overline{E}[\|(F-G)\cdot S\|_\infty 1_{\{\langle S\rangle_T \le c\}} 1_{\{\|(F-G)\cdot S\|_\infty \le ab\sqrt{c}\}}] \\
&\le 2\exp\left(-\frac{b^2}{2d}\right) + \frac{d_c(F,G)}{a} + ab\sqrt{c}
\end{aligned}
$$

Conclusion of Model Free Ito Integration

Theorem

Let F be an adapted, cadlag process with values in $\mathbb{R}^d$. Then there exists $\int FdS \in \overline{L}_0([0,T],\mathbb{R})$ s.t. for every sequence of step function (F^n) satisfying $\lim_{n\to\infty} d_\infty(F^n, F) = 0$ we have $\lim_{n\to\infty} d_{loc}((F^n \cdot S), \int FdS) = 0$.
The integral process $\int FdS$ is continuous for typical price paths and there is a representative which is adapted, although it takes valued in $\overline{\mathbb{R}}$. The map $F \mapsto \int FdS$ is linear and satisfies

$$d_{loc}\left(\int FdS, \int GdS\right) \lesssim d_\infty(F,G)^{1/2-\varepsilon} \tag{17}$$

for every $\varepsilon > 0$.

Rough Path Integration for Typical Price Paths

Definition

Let $p \in (2,3)$. A p-rough path is a map $\mathbb{S} = (S, A) : \Delta_T \to \mathbb{R}^d \times \mathbb{R}^{d \times d}$ s.t.

ANA $[S]_{p-var} + [A]_{p/2-var} < \infty$

CHEN $A_{ij}(s,t) = A_{ij}(s,u) + A_{ij}(u,t) + S_i(s,u)S_j(u,t)$

Definition

Let $p \in (2,3)$ and $q > 0$ s.t. $\frac{2}{p} + \frac{1}{q} > 1$. Let $\mathbb{S} = (S, A)$ be a p-rough path, $F : [0,T] \to \mathbb{R}^n$, and $F' : [0,T] \to \mathbb{R}^{n \times d}$. A pair (F, F') is **controlled** by S if the derivative F' has finite q-variation and the remainder $R_F : \Delta_T \to \mathbb{R}^n$, defined by

$$R_F(s,t) := F_{s,t} - F'_s S_{s,t} \tag{18}$$

has finite r-variation with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Denote collection of those (F, F') by $\mathscr{C}^q_{\mathbb{S}}$, together with the semi-norm

$$\|(F, F')\|_{\mathscr{C}^q_{\mathbb{S}}} := \|F'\|_{q-var} + \|R_F\|_{r-var} \tag{19}$$

The norm $|F_0| + |F'_0| + \|(F, F')\|_{\mathscr{C}^q_{\mathbb{S}}}$ makes $\mathscr{C}^q_{\mathbb{S}}$ into a Banach space.

Theorem

Let $p \in (2,3)$ and $q > 0$ s.t. $\frac{2}{p} + \frac{1}{q} > 1$. Let $\mathbb{S} = (S, A)$ be a p-rough path and $(F, F') \in \mathscr{C}^q_{\mathbb{S}}$. Then there exists a unique function $\int F dS \in C([0,T], \mathbb{R}^n)$ which satisfies

$$\left| \int_s^t F_u dS_u - F_s S_{s,t} - F'_s A(s,t) \right| \tag{20}$$

$$\lesssim \|S\|_{p-var,[s,t]} \|R_F\|_{r-var,[s,t]} + \|A\|_{p/2-var,[s,t]} \|F'\|_{q-var,[s,t]} \tag{21}$$

for every $(s,t) \in \Delta_T$. Furthermore, we have

$$\int_0^t F_u dS_u = \lim_{|P| \downarrow 0} \sum_{s,r \in P} \left[F_s S_{s,r} + F'_s A(s,r) \right] \tag{22}$$

for any partition with mesh going to 0.

Proposition

Let $p \in (2,3)$ *and* $q > 0$ *s.t.* $\frac{2}{p} + \frac{1}{q} > 0$. *Let* $\mathbb{S} = (S, A)$ *and* $\tilde{\mathbb{S}} = (\tilde{S}, \tilde{A})$ *be two* p*-rough paths, and let* $(F, F') \in \mathscr{C}^q_{\mathbb{S}}$ *and* $(\tilde{F}, \tilde{F}') \in \mathscr{C}^q_{\tilde{\mathbb{S}}}$. *Then for every* $M > 0$ *there is a* $C_M > 0$ *s.t.*

$$\left\| \int_0^{\cdot} F_s dS_s - \int_0^{\cdot} \tilde{F}_s d\tilde{S}_s \right\|_{p-var}$$
$$\leq C_M \big(|F_0 - \tilde{F}_0| + |F'_0 - \tilde{F}'_0| + \|F' - \tilde{F}'\|_{q-var}$$
$$+ \|R_F - R_{\tilde{F}}\|_{r-var} + \|S - \tilde{S}\|_{p-var} + \|A - \tilde{A}\|_{p/2-var} \big)$$

$\Rightarrow$ rough integration is locally Lipschitz

Theorem

For $(s,t) \in \Delta_T$, $\omega \in \Omega$ *and* $1 \leq i,j \leq d$ *define*

$$A^{i,j}_{s,t}(\omega) := \int_s^t S^i_r dS^j_r(\omega) - S^i_s(\omega) S^j_{s,t}(\omega) := \int_0^t S^i_r dS^j_r(\omega) - \int_0^s S^i_r dS^j_r(\omega) - S^i_s(\omega) S^j_{s,t}(\omega) \tag{23}$$

where $\int S^i dS^j$ *is the the model free Ito integral. If* $p > 2$*, then for typical price paths* $A = (A^{i,j})_{1 \leq i,j \leq d}$ *has finite* $p/2$*-variation, and in particular* $\mathbb{S} = (S, A)$ *is a* p*-rough path.*

Recap & Discussion

Conclusion

- gives approximation of rough path integral through **non**-anticipating, **non**-compensated Riemann sums
- does not give explicit formulas for (say) the price of an option
- The integral $\int_0^t F_s dS_s$ involves the values of S all the way up to t. If those are not observable, what to do?

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