On the Yoneda Lemma in Category Theory

G. Chiusole

Abstract

The following essay shall briefly explore the Yoneda Lemma as well as some corollaries and its implications and interpretations. In essence, the Yoneda Lemma asserts an important correspondence between the set of natural transformations of two functors $\mathfrak{h}^A, \mathfrak{f} \in \mathtt{Nat}(\mathbf{C}, \mathbf{Set})$, and the set $\mathfrak{f}(A)$. Through this one can completely characterize such a functor via an element in the set $\mathfrak{f}(A)$ - a generic element - providing a tool through which highly complex functors can be represented via (comparably) simple elements and morphisms of sets. It is simultaneously a powerful lemma in the construction of isomorphism theorems; most famously the Yoneda Embedding.

1 Introduction

Even though the Yoneda Lemma can be interpreted in a purely algebraic way [Pratt, 2009], we want to study it in the way it was originally developed, i.e. in the context of Category Theory (CT). The development of CT in its axiomatic form began shortly after World War II and was initiated by S. Mac Lane and S. Eilenberg in their study of algebraic topology. In a nutshell, a category is a collection of *objects* and *morphisms* between them requiring the existence of composition of those morphisms - that is, of course, subjected to conditions. CT is, fundamentally, the study of such categories, but more importantly the study of morphisms: inside categories, between them (aka. functors), between functors (aka. natural transformations), ... Named after the Japanese pioneer N. Yoneda, the lemma was developed in the second half of the 20^{th} century. It is arguably one of the, if not the single most central theorem in the theory of categories.
The original motivation for CT was to understand algebraic structures through the structure preserving maps between them - the homomorphisms - an idea already pursued by E. Noether in her study of rings in the beginning of the 20th Century. In this sense, CT may be viewed as an external view to algebra, as opposed to the internal, constructive nature of set theory. The Yoneda Lemma in this sense, provides a bridge between this external and internal view.

2 Yoneda Lemma

Let us start by formally stating the lemma in question:

Lemma. *Let* $\mathbf{C}$ *be a locally small category*[1] *and let* $\mathbf{Set}$ *be the category of sets. Let* $\mathfrak{h}^A : \mathbf{C} \longrightarrow \mathbf{Set}$ *be a hom-functor on* $\mathbf{C}$ *based at the object* $A \in \mathbf{C}$. *Let* $\mathfrak{f} : \mathbf{C} \longrightarrow \mathbf{Set}$ *be a set-valued covariant functor on* $\mathbf{C}$. *Then the set of morphisms between the two functors* $\mathfrak{h}^A$ *and* $\mathfrak{f}$ *stands in natural isomorphism to the set* $\mathfrak{f}A$ *which is assigned to* A *by* $\mathfrak{f}$. *Hence there exists a natural isomorphism* $\Gamma : \mathtt{Nat}(\mathbf{C}, \mathbf{Set})(\mathfrak{h}^A, \mathfrak{f}) \longrightarrow \mathfrak{f}A$.

Proof. $\Gamma : \eta \mapsto \eta_A(id_A)$ is such an isomorphism. In order not to disrupt the flow of this article and to keep it concise, we will omit a formal proof of this lemma, and instead present the following diagram, examining the components of η at A. Via diagram chasing we can show that Γ is indeed injective and natural, and by constructing explicit natural transformations $\mathfrak{h}^A A \longrightarrow \mathfrak{f}A$ we can show its surjectivity, proving Γ to be a natural isomorphism.

$$\begin{array}{ccccccc}
A & \quad & \mathfrak{h}^A A = \mathrm{Hom}_{\mathbf{C}}(A,A) & \xrightarrow{\eta_A} & \mathfrak{f}A & \quad & id_A \longmapsto^{\eta_A} \eta_A(id_A) \\
a \downarrow & & \mathfrak{h}^A a \downarrow & & \downarrow \mathfrak{f}a & & a \circ - \downarrow \qquad \mathfrak{f}a \downarrow \\
B & & \mathfrak{h}^A B & \xrightarrow{\eta_B} & \mathfrak{f}B & & a \circ id_A = a \longmapsto^{\eta_B} \eta_B(a) = \mathfrak{f}a(\eta_A(id_A))
\end{array}$$

A formal proof can be found in [Hedman, 2016]. □

[1] From here on every category is understood to be locally small.

3 Interpretation and Corollaries

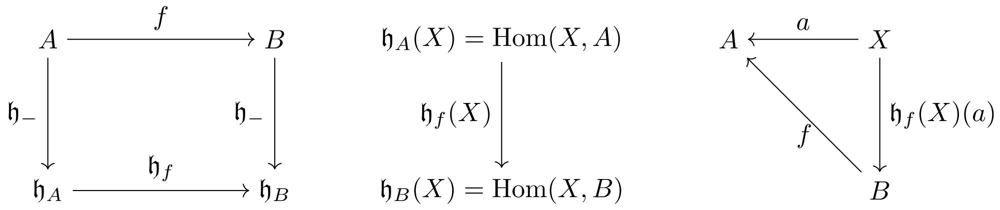
Undoubtedly, the lemma has its greatest application in the rather more abstract and advanced areas of mathematics, such as group cohomology, topos theory and algebraic topology in general - certainly subjects I do not feel comfortable talking about as if I understood them. Hence, let me offer much more simple applications of it:

3.1 "Tell me who your friends are and I'll tell you who you are." - Yoneda Embedding

Abstractly, we now want to construct another functor that arises very naturally in exploring hom functors and related notions: In particular, a functor that assigns to an object A in a category $\mathbf{C}$ its corresponding hom-functor $\mathfrak{h}^A$, who, in turn assigns to other objects X the hom-set $\mathrm{Hom}(A, X)$, hence we want this functor to have the defining property $A \mapsto \mathfrak{h}^A$. By the definition of $\mathfrak{h}^A$, the image of $\mathfrak{h}^-$ is the functor category $\texttt{Funct}[\mathbf{C}, \mathbf{Set}]$. Having said that, with our application in mind, we will immediately proceed to construct the dual functor $\mathfrak{h}_- : \mathbf{C} \longrightarrow \texttt{Funct}[\mathbf{C}^{op}, \mathbf{Set}]$ instead.
We define as follows

$$\mathfrak{h}_- : \mathbf{C} \longrightarrow \texttt{Funct}[\mathbf{C}^{op}, \mathbf{Set}] \tag{1}$$
$$A \mapsto \big(\mathfrak{h}_A : X \mapsto \mathrm{Hom}(X, A)\big) \ . \tag{2}$$

The definition of $\mathfrak{h}_-$ acting on a morphism $f : A \longrightarrow B$ follows by observing this diagram in the category of sets. In the second (middle) diagram we study $\mathfrak{h}_A$ and $\mathfrak{h}_B$ by their action on arbitrary object in $\mathbf{C}$ (on an object of their domain).



Hence $\mathfrak{h}_f(X) : a \mapsto f \circ a$.
The definition of $\mathfrak{h}_f$ following naturally from $\mathfrak{h}_X$ leads to the fact that $\mathfrak{h}_-$ is co-variant - hence $\mathfrak{h}_- : \mathbf{C} \longrightarrow \texttt{Funct}[\mathbf{C}^{op}, \mathbf{Set}]$. Indeed, if we want to consider a concrete $\mathfrak{h}_X$ to be contra-variant, the functor whose image $\mathfrak{h}_X$ needs to be co-variant. Of course, the opposite is true for the dual of the functor $\mathfrak{h}^- : X \mapsto \mathfrak{h}^X$. This forces $\mathfrak{h}^-$ to be a functor $\mathbf{C}^{op} \longrightarrow \texttt{Funct}[\mathbf{C}, \mathbf{Set}]$

Theorem. *The covariant Yoneda Functor $\mathfrak{h}_-$ is full and faithful and thus an embedding of $\mathbf{C}$ in the corresponding category of pre-sheaves $\texttt{Funct}[\mathbf{C}^{op}, \mathbf{Set}]$. Hence $\mathbf{C} \hookrightarrow \texttt{Funct}[\mathbf{C}^{op}, \mathbf{Set}]$.*

Proof. Consider the Yoneda Lemma, defining $\mathfrak{f} := \mathfrak{h}_B$. Then $\texttt{Nat}(\mathbf{C}^{op}, \mathbf{Set})(\mathfrak{h}_A, \mathfrak{h}_B) \simeq \mathrm{Hom}(A, B)$. Hence the natural transformation between the functors $\mathfrak{h}_A$ and $\mathfrak{h}_B$ are in one-to-one correspondence with the morphisms between the objects A and B. By letting $\mathfrak{h}_A$ and $\mathfrak{h}_B$ act on an object $X \in \mathbf{C}$, we obtain $\big(\mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, B)\big) \simeq \mathrm{Hom}(A, B)$. $\mathfrak{h}_-$ is obviously a bijection w.r.t. the objects in $\mathbf{C}$. In conjunction with the above diagram it is now clear that $\mathfrak{h}_-$ is also bijective w.r.t. the morphisms. Thus $\mathfrak{h}_-$ is a bijection in general, and hence an embedding. □

The same is true for the theorem regarding the contra-variant Yoneda functor $\mathfrak{h}^- : \mathbf{C}^{op} \longrightarrow \texttt{Funct}[\mathbf{C}, \mathbf{Set}]$. Notice though the reversion of morphisms due to the contra-variance of $\mathfrak{h}^-$.
From this we see that an object A is fully determined by its relation to other objects, that is, its hom-sets to them. For example, in the category of sets $\mathbf{Set}$ every set A is entirely determined by its hom-set $\mathrm{Hom}(\mathbb{1}, A)$ with the terminal object $\mathbb{1}$. In other categories this characterization might not work with a single hom-set. Consider for example the category of groups $\mathbf{Grp}$, where $\forall G_1, G_2 : \mathrm{Hom}(\mathbb{1}, G_1) = \mathrm{Hom}(\mathbb{1}, G_2) = \{1_{\mathbb{1}} \mapsto 1_G\}$, with 1 being the identity element w.r.t. the group operation. The Yoneda Embedding guarantees, however, that (at least) all of the hom-sets give enough data to recover A.
In the words of B. Mazur, "Thinking about Grothendieck" (qtd. in [Riehl, 2017])

> [...] a mathematical object X is best thought of in the context of a category surrounding it, and is determined by the network of relations it enjoys with all the objects of that category. Moreover, to understand X it might be more germane to deal directly with the functor representing it.

3.2 Representation of Functors and Presheaves

Another more concrete interpretation/application comes by the following corollary. With the upcoming example in mind, we choose the co-variant version of the corollary. First, recall the definition of a representation of a functor $\mathfrak{f} : \mathbf{C} \longrightarrow \mathbf{Set}$ on $\mathbf{C}$ to be a pair $(A \in \mathbf{C}, \alpha : \mathfrak{h}^A \longrightarrow \mathfrak{f})$ consisting of an object of the underlying category and a natural isomorphism from the co-variant hom-functor based at that object to the functor $\mathfrak{f}$. [Barr und Wells, 2000]

Corollary. *For every representation $(A \in \mathbf{C}, \alpha : \mathfrak{h}^A \longrightarrow \mathfrak{f})$, there uniquely exists a pair $(A \in \mathbf{C}, x \in \mathfrak{f}(A))$ s.t. the following diagram commutes*

$$
\begin{array}{c}
\mathbf{C} \\
\Big\downarrow{\scriptstyle \mathfrak{f}} \\
\mathbf{Set}
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\;a\;} & B \\
{\scriptstyle \mathfrak{f}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{f}} \\
\mathfrak{f}A & \xrightarrow{\;\mathfrak{f}a\;} & \mathfrak{f}B
\end{array}
\qquad
\begin{array}{ccc}
x & \stackrel{a}{\longmapsto} & y \\
{\scriptstyle \mathfrak{f}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{f}} \\
u & \stackrel{\mathfrak{f}a}{\longmapsto} & v
\end{array}
$$

Hence $(\mathfrak{f}a)u = v$.

Proof. Consider one particular $\alpha : \mathfrak{h}^A \longrightarrow \mathfrak{f}$. The result follows. ☐

There exist thus two distinct ways of representing functors and pre-sheaves. We will elaborate with the following example. [Leinster, 2000]

Example. Consider the category of vector spaces **Vec**, the two objects $U, V \in \mathbf{Vec}$ and the co-variant functor

$$\mathfrak{b}^{U,V} : \mathbf{Vec} \longrightarrow \mathbf{Set} \tag{3}$$
$$W \mapsto \mathrm{BiLin}(U, V; W) \ . \tag{4}$$

We can represent this functor $\mathfrak{b}^{U,V}$ as one of the following:

(a) A vector space $T \in \mathbf{Vec}$ and a natural isomorphism $\alpha : \mathfrak{h}^T \longrightarrow \mathfrak{b}^{U,V}$. In more familiar terms this is the vector space ("tensor space") T and the bijection $\alpha : \mathbf{Vec}(T, W) \simeq \mathbf{Vec}(U, V; W)$.

(b) A vector space $T \in \mathbf{Vec}$ and an element of $\mathfrak{b}^{U,V}(W) = \mathrm{BiLin}(U, V; W)$. Hence, a bi-linear map $h \in \mathbf{Vec}(U, V; T) : U \times V \longrightarrow T$ s.t. the following diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\qquad a \qquad} & S \\
{\scriptstyle \mathfrak{f}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{f}} \\
\underbrace{t \in \mathfrak{b}^{U,V} T}_{=\mathrm{BiLin}(U,V;T)} & \xrightarrow{\;\mathfrak{f}a\;} & \underbrace{s \in \mathfrak{b}^{U,V} S}_{=\mathrm{BiLin}(U,V;S)}
\end{array}
\qquad \text{hence} \qquad
\begin{array}{ccc}
U \times V & \xrightarrow{\;t\;} & T \\
& {\scriptstyle s}\searrow & \Big\downarrow{\scriptstyle \mathfrak{f}a} \\
& & S
\end{array}
$$

commutes and thus $s \mapsto \mathfrak{f}a \circ s$.

After all we can sum up the importance of the Yoneda Lemma by the following joke:

A student jokingly asks: "How do you put an elephant into a refrigerator?"
"Isn't this just a special case of the Yoneda Lemma?" answers the Category Theorist.[Smith, 2010]

References

[Barr und Wells 2000] Barr, Michael ; Wells, Charles: Toposes, Triples, and Theories. (2000), S. 23–24

[Hedman 2016] Hedman, Jonas: 2-Categories and Yoneda lemma. (2016), S. 42–46

[Leinster 2000] Leinster, Tom: The Yoneda Lemma: What's It All About? (2000), S. 6–7

[Pratt 2009] Pratt, Vaughan: The Yoneda Lemma without Category Theory: Algebra and Applications. (2009)

[Riehl 2017] Riehl, Emily: *Category Theory in Context*. Courier Dover Publications, 2017

[Smith 2010] Smith, Robert: *A Joke*. 2010. – URL `http://symbo1ics.com/blog/?p=389`. – Zugriffsdatum: 2018-06-07