Gaussian Measures in Hilbert Spaces [Da Prato, 2006, Chap. 1]

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Motivation

"There is no infinite dimensional Lebesgue measure"

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Theorem

Let $(E, \| \cdot \|)$ *be a normed space with* $\dim E = \infty$. *Then there is no non-trivial, translation-invariant,* σ*-additive Borel measure* μ *on* $(E, \| \cdot \|)$ *s.t.* $\mu[B_\varepsilon(0)] < \infty$ *for all* $\varepsilon > 0$.

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Alternative: Gaussian measures

Notation, Review, etc.

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Unless otherwise specified, H denotes a real, separable Hilbert space.

$$\mathcal{B}(H) \text{... Borel } \sigma - \text{algebra on } H$$

$$L(H) := \{T \in L(H) | \text{ linear, bounded}\}$$

$$L^+(H) := \{T \in L(H) | \text{ symmetric, pos. semi-definite}\}$$

$$L_1^+(H) := \left\{ T \in L^+(H) \middle| \sum_{k=1}^{\infty} \langle e_k, Te_k \rangle < \infty, (e_k)_{k \in \mathbb{N}} \text{ ONB of } H \right\}.$$

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For $H = \mathbb{R}^d$ we have

$$L_1^+(\mathbb{R}^d) = L^+(\mathbb{R}^d) \subseteq L(\mathbb{R}^d).$$

Spectral Theorem for $Q \in L_1^+(H)$

Theorem

Let $Q \in L_1^+(H)$. Then there exists an ONB $(e_k)_{k\in\mathbb{N}}$ of H and a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^1$ s.t.

$$\forall k \in \mathbb{N} : Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0$$

in particular $\lambda_k \to 0$ as $k \to \infty$.

Product measures

Define

$$\mathcal{F} := \sigma\Big(\underbrace{\{x \in \mathbb{R}^\infty : (x_{k_1}, \ldots, x_{k_n}) \in A\}}_{C_{k_1,\ldots,k_n,A}} : A \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}\Big)$$

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Proposition

$\mathcal{F}$ coincides with $\mathcal{B}(\mathbb{R}^\infty)$ and the σ-algebra generated by the projections.

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Theorem

Let $(\mathbb{P}_k)_{k\in\mathbb{N}}$ be a sequence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then there exists a unique probability measure on $(\mathbb{R}^\infty, \mathcal{F})$ s.t. for every $C_{k_1,\dots,k_n,A}$ we have

$$\mathbb{P}(C_{k_1,\dots,k_n,A}) = (\mathbb{P}_{k_1} \times \ldots \times \mathbb{P}_{k_n})(A).$$

In particular, for every $i \in \mathbb{N}$ the projection onto the k-th coordinate $\pi_k : x \mapsto x_k$ has distribution $\mathbb{P}_k$ and $\{\pi_k\}_{k=1}^\infty$ is a set of independent real valued random variables w.r.t. $\mathbb{P}$.

One-dimensional Hilbert spaces

Definition (1-dim.)

Let $a \in \mathbb{R}, \lambda \geq 0$. Then define the measure $N_{a,\lambda}$ on $\mathcal{B}(\mathbb{R})$ by

$$(\lambda = 0) \quad N_{a,\lambda}(B) = \delta_a(B) = \begin{cases} 1 & a \in B, \\ 0 & a \notin B \end{cases}, \quad \forall B \in \mathcal{B}(\mathbb{R}),$$

and

$$(\lambda \neq 0) \quad \mathrm{d}N_{a,\lambda}(x) = \frac{1}{\sqrt{2\pi\lambda}} \exp\left\{-\frac{(x-a)^2}{2\lambda}\right\} \mathrm{d}x.$$

Moments and characteristic function (1-dim.)

For $a \in \mathbb{R}, \lambda \geq 0$ we have

mean	a	$=$	$\int_{\mathbb{R}} x \,\mathrm{d}N_{a,\lambda}(x)$
variance	λ	$=$	$\int_{\mathbb{R}} (x-a)^2 \,\mathrm{d}N_{a,\lambda}(x)$
char. function	$\widehat{N_{a,\lambda}}(h)$	$=$	$\exp\left\{iah - \frac{1}{2}\lambda h^2\right\}, \quad h \in \mathbb{R}$

Finite-dimensional Hilbert spaces

Definition (fin. dim.)

Definition

A measure μ on H is called Gaussian, if for every $h \in H$ the functional $x \mapsto \langle h, x \rangle$ has law $N_{a,\lambda}$ for some $a \in \mathbb{R}, \lambda \geq 0$.

Construction (fin. dim.)

1. Let H be a real Hilbert space with $\dim(H) = d$, $a \in H$, $Q \in L^+(H)$. Then let $\{e_1, \dots, e_d\} \subseteq H$ be an ONB of H s.t.

$$\forall 1 \leq k \leq d : Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0 \ .$$

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2. Identify H with $\mathbb{R}^d$ via $x \mapsto (\langle x, e_1 \rangle, \ldots, \langle x, e_d \rangle)$.
3. Then define the measure $N_{a,Q}$ on $\mathcal{B}(H)$ by

$$N_{a,Q} = \bigtimes_{k=1}^{d} N_{a_k, \lambda_k}$$

Is this construction Gaussian?

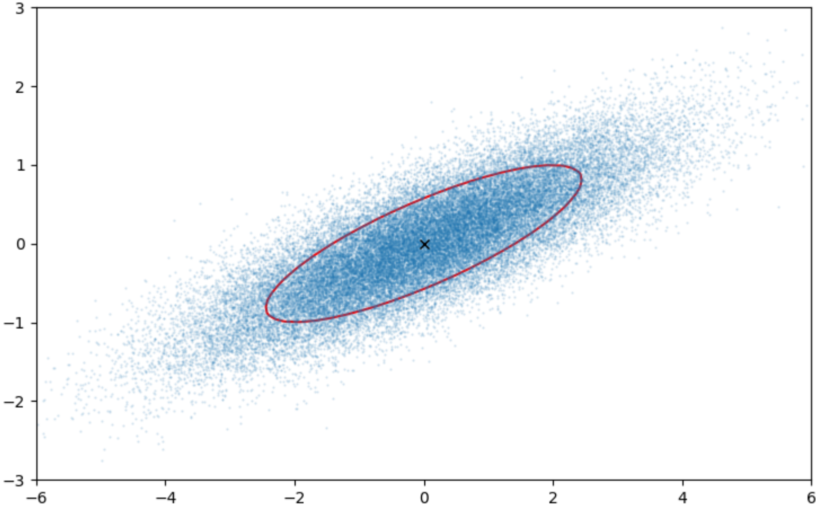
Theorem

$N_{a,Q}$ *is a Gaussian measure.*

Proof.

□

Figure in $H = \mathbb{R}^2$



Moments and characteristic function (fin. dim.)

For $a, \in H, Q \in L^+(H)$ we have

mean	a	$=$	$\int_H x \, \mathrm{d}N_{a,Q}(x)$
covariance	$\langle y, Qz \rangle$	$=$	$\int_H \langle y, (x-a) \rangle \langle z, (x-a) \rangle \, \mathrm{d}N_{a,Q}(x)$
char. functional	$\widehat{N_{a,Q}}(h)$	$=$	$\exp\left\{ i\langle h, a \rangle - \frac{1}{2}\langle h, Qh \rangle \right\}, \quad h \in H$

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Proposition

If $\det(Q) > 0$ *i.e.* $\lambda_k > 0$ *for every* $k \in \{1, \dots, d\}$*, then*

$$\mathrm{d}N_{a,Q}(x) = \frac{1}{\sqrt{(2\pi)^d \det Q}} \exp\left\{ -\frac{1}{2} \left\langle (x-a), Q^{-1}(x-a) \right\rangle \right\} \mathrm{d}x.$$

Separable Hilbert spaces

Definition of mean

Let μ be a measure on $(H, \mathcal{B}(H))$ s.t. $\int_H \|x\| \, \mathrm{d}\mu(x) < \infty$.

Then $h \mapsto F(h) := \int_H \langle h, x \rangle \, \mathrm{d}\mu(x)$ is bounded since

$$|F(h)| \leq \int_H |\langle h, x \rangle| \, \mathrm{d}\mu(x) \leq \|h\| \underbrace{\int_H \|x\| \, \mathrm{d}\mu(x)}_{<\infty}$$

Thus by Riesz' Representation theorem $\exists! a \in H$:

$$\langle h, a \rangle = \int_H \langle h, x \rangle \, \mathrm{d}\mu(x), \quad h \in H.$$

called the **mean of** μ.

Intermezzo: Bochner spaces

It is clear how to integrate $f : H \to \mathbb{R}$ when there is a measure on H. But how about $f : H \to H$, e.g. $x \mapsto x$, as in the definition of the mean?

Intermezzo: Bochner spaces

It is clear how to integrate $f : H \to \mathbb{R}$ when there is a measure on H. But how about $f : H \to H$, e.g. $x \mapsto x$, as in the definition of the mean?

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Then define the Bochner space

$$L^p(\Omega; H) := \left\{ u : \Omega \to H | u \text{ measurable}, \underbrace{\int_\Omega \|u(\omega)\|_H^p \, \mathrm{d}\mathbb{P}(\omega)}_{=:\|u\|_{L^p(\Omega;H)}^p} < \infty \right\}$$

where $1 \leq p < \infty$.

Proposition

The set $\left\{ \sum_{i=1}^{n} 1_{A_i} h_i : A_i \in \mathcal{F}, h_i \in H \right\}$ *of simple functions lies dense in* $(L^p(\Omega; H), \|\cdot\|_{L^p(\Omega;H)})$.

Intermezzo: Bochner integral

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Definition

For $\sum_{i=1}^n 1_{A_i} h_i \in L^1(\Omega; H)$ define

$$\int \sum_{i=1}^n 1_{A_i} h_i \mathrm{d}\mathbb{P} = \sum_{i=1}^n \mathbb{P}(A_i) h_i \in H$$

For $u \in L^1(\Omega; H)$, define the Bochner integral of u as

$$\int u \ \mathrm{d}\mathbb{P} := \lim_{k \to \infty} \int \sum_{i=1}^{n^{(k)}} 1_{A_i^{(k)}} h_i^{(k)} \mathrm{d}\mathbb{P} \in H$$

Proposition

Let $f : H \to \mathbb{R}$ *be a bounded linear functional and* $u \in L^1(\Omega; H)$*. Then*

$$f\left[\int u(\omega)\ \mathrm{d}\mathbb{P}(\omega)\right] = \int f\left[u(\omega)\right]\ \mathrm{d}\mathbb{P}(\omega)$$

Characterization of the mean

Theorem

Indeed,

$$a = \int_H x \, \mathrm{d}\mu(x).$$

Proof.

Let $a \in H$ be the mean of μ and let $h \in H$ be arbitrary. Then $x \mapsto \langle h, x \rangle$ defines a bounded linear functional on H. Hence it can be pulled into the integral and we have

$$\langle h, a \rangle = \int_H \langle h, x \rangle \, \mathrm{d}\mu(x) = \left\langle h, \int_H x \, \mathrm{d}\mu(x) \right\rangle$$

Uniqueness of a gives the result. $\square$

Definition the covariance

Let μ be a measure on $(H, \mathcal{B}(H))$ s.t. $\int_H \|x\|^2 \, \mathrm{d}\mu(x) < \infty$.

Then $(h,k) \mapsto G(h,k) := \int_H \langle h, x-a\rangle \langle k, x-a\rangle \, \mathrm{d}\mu(x)$ is bounded since

$$\begin{aligned} |G(h,k)| &\leq \int_H |\langle h, x-a\rangle| \, |\langle k, x-a\rangle| \, \mathrm{d}\mu(x) \\ &\leq \|h\| \, \|k\| \underbrace{\int_H \|x-a\|^2 \, \mathrm{d}\mu(x)}_{<\infty} \end{aligned}$$

Thus by Riesz' Representation theorem there exists a unique bounded linear operator $Q : H \to H$ s.t.

$$\langle h, Qk\rangle = \int_H \langle h, x-a\rangle \langle k, x-a\rangle \, \mathrm{d}\mu(x), \quad h, k \in H.$$

called the **covariance of** μ.

Properties of the covariance

Theorem

Let μ be a measure on $(H, \mathcal{B}(H))$ s.t. a and Q exist. Then $Q \in L_1^+(H)$ i.e. Q is symmetric, positive semi-definite and of trace class.

Proof.

$\square$

Definition of Gaussian Measures

Definition

A measure μ on $(H, \mathcal{B}(H))$ is called Gaussian if $\exists a \in H, Q \in L_1^+(H)$ s.t.

$$\int_H \exp\{i\langle h, x\rangle\} \, \mathrm{d}\mu(x) = \underbrace{\exp\left\{i\langle a, h\rangle - \frac{1}{2}\langle h, Qh\rangle\right\}}_{=:\widehat{N_{a,Q}}(h)}, \quad h \in H.$$

$N_{a,Q}$ is called **non-degenerate** if $\ker Q = \{0\}$.

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Recall: for $H = \mathbb{R}^n$ the Fourier inversion theorem asserts that two measures with the same characteristic functional are equal. This also is still true when $\dim H = \infty$. In particular, Gaussian measures are entirely characterized by their mean and covariance operator.

Existence of Gaussian measures

1. Let $a \in H$ and $Q \in L^+(H)$ with $\{e_k\}_{k\in\mathbb{N}} \subseteq H$ an ONB of H associated to Q. Then

$$\forall k \in \mathbb{N} : Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0 \quad .$$

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2. Identify H with ℓ^2 via $x \mapsto (\langle x, e_k\rangle)_{k\in\mathbb{N}}$.
3. Define the measure $N_{a,Q}$ on $\mathcal{B}(\mathbb{R}^\infty)$ by

$$N_{a,Q} = \bigtimes_{k\in\mathbb{N}} N_{a_k,\lambda_k}.$$

Definition (separable)

This gives a measure on $\mathbb{R}^\infty := \bigtimes_{k\in\mathbb{N}} \mathbb{R}$ and not on ℓ^2, but

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Theorem

$\mu := N_{a,Q}$ *is concentrated on* ℓ^2 *i.e.* $\mu(\ell^2) = 1$.

Proof.

□

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Theorem

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Proof.

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Closing remarks

What if H is less well-behaved?

More generally, how can one define a Gaussian measure?

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More generally, how can one define a Gaussian measure?

1. via a density (needs $\dim H < \infty$)
2. via cont. linear functions (needs rich enough dual theory e.g. loc. convex TVS)
3. via the characteristic functional (needs Fourier theory on H)
4. via identification $H \simeq \ell^2$ (needs H to be a separable Hilbert space)

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