Maximal Tori I

Gideon Chiusole

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Throughout these notes, unless specified otherwise, we make the following conventions. G denotes a compact and connected Lie group. For a smooth map $\varphi : M \to N$ between smooth mainfolds φ_* or $D\varphi$ denotes the pushforward/differential, while φ^* denotes the pullback. The tangent space at the identity will be denoted by LG (if G is not a Lie group, LG denotes the tangent space at a distinguished point which will be clear from the context). Conjugation by an element of $g \in G$ will be denoted by $c(g)$, while the image of $g \in G$ under the adjoint representation will be denoted by $\mathrm{Ad}(g)$.

1. Tori and the Weyl Group

1.1. Abelian Lie Groups

Definition 1.1. A torus T in G is a compact, connected, abelian immersed Lie subgroup. A torus T in G is called **maximal** if, for any other torus T' we have

$$T \subseteq T' \Rightarrow T = T'. \tag{1}$$

Observation 1.2. *By (a consequence of) Cartan's theorem a torus is an embedded subgroup.*

Proof. Let $\iota : T \hookrightarrow G$ denote the inclusion of the Lie subgroup. Since T is compact, so is $i(T)$. Thus $i(T) \subseteq G$ is a closed subgroup and thus by Cartan's theorem i is an embedding of Lie groups. □

Observation 1.3. *G contains a maximal torus T. However, T is in general (and usually) not unique. In fact, we will see later (Theorem 2.1) that if the maximal torus is unique, then it coincides with G.*

Proof. Let $\mathfrak{T}(G)$ denote the set of tori in G. The 0-torus $T^0 = \{e\} \subseteq G$ is a Lie subgroup which is a torus, i.e. $\mathfrak{T}(G) \neq \emptyset$. Assume not that $\mathfrak{T}(G)$ did not have a maximal element. Then for any $T \in \mathfrak{T}(G)$ there exists a $T' \neq T$ s.t. $T \subsetneq T' \subseteq G$. Since tori are compact and connected, Lemma A.1 implies that

$$\dim(T) < \dim(T') \leq \dim(G). \tag{2}$$

Since $\dim(G) < \infty$, this is a contradiction. □

As the name suggests, a torus in the sense of Definition 1.1 is isomorphic to a torus in the usual sense i.e. $\mathbb{R}^d / \mathbb{Z}^d$ for some $d \geq 0$. This follows from the classification of abelian Lie groups:

Theorem 1.4 (Classification of abelian Lie groups; [BtD, I. (3.6), (3.7)])**.**

1. *Let G be a connected, abelian Lie group. Then $G \cong \mathbb{T}^d \times \mathbb{R}^e$ for some $d, e \geq 0$.*
2. *Let G be a compact, abelian Lie group. Then $G \cong \mathbb{T}^d \times \prod_{i=1}^k \mathbb{Z} / n_i \mathbb{Z}$ for some $d, n_1, \ldots, n_k \geq 0$.*

Definition 1.5. Let T be a maximal torus in G, and let

$$N := \{g \in G : gTg^{-1} = T\} \tag{3}$$

be the normalizer of T in G. The the group $W := N/T$ is called the **Weyl group** of G.

Note that since T is a closed subgroup, so is

$$N = (c(\cdot)(t))^{-1}(T) \cap \bigcap_{t \in T} (c((\cdot)^{-1})(t))^{-1}(T), \tag{4}$$

where $g \mapsto c(g)(t) = gtg^{-1}$ denotes conjugation by g, applied to t.
By Definition 1.5, for a given G, the Weyl group W depends on the maximal torus T in G. However, as will be shown in Theorem 2.1 all maximal tori are conjugate to each other and as a consequence all Weyl groups are isomorphic.

The normalizer N operates on T via

$$N \times T \to T; \quad (n, t) = ntn^{-1} \tag{5}$$

and, since T is abelian and thus operates trivially on itself, the Weyl group also operates on T via

$$W \times T \to T; \quad (nT, t) = ntn^{-1}. \tag{6}$$

Theorem 1.6. *The Weyl group W of G is finite.*

Proof. Let N_0 denote the connected component of N containing the identity[1]. We will show that $N_0 \subseteq T$ and thus $N_0 = T$. It then follows that 1) since N is compact, so is $W = N/T = N/N_0$, and 2) since, as the homeomorphic image of an open set, nN_0 is open in N for every $n \in N$, and $[n] \in N/N_0$ is open precisely if $\pi^{-1}(nN_0) = nN_0 \subseteq N$ is open, that every singleton in W is open. Hence W is 1) compact and 2) discrete and thus finite.

Now, to see that $N_0 \subseteq T$ let $k := \dim(T)$. Recall that the automorphisms of a torus T are precisely those linear transformations on $\mathbb{R}^d$ that preserve the lattice $\mathbb{Z}^d$, i.e. $\mathrm{Aut}(T) = \mathrm{GL}(k, \mathbb{Z}) \subseteq \mathrm{GL}(k, \mathbb{R})$ and consider the continuous map

[1] Recall that the connected component of a topological group is necessarily a subgroup: Since $\cdot : G_0 \times G_0 \to G$ is continuous and G_0 is connected, the image is connected. Thus, since $e = e \cdot e \in \cdot(G_0 \times G_0)$, the image is contained in a connected component, which contains the identity, i.e. it is contained in G_0. The same argument works for $(\cdot)^{-1} : G_0 \to G$.

$$N \xrightarrow{c} \mathrm{Aut}(T) \xrightarrow{D} \mathrm{Aut}(LT) \cong \mathrm{Aut}(\mathbb{R}^k) \cong GL(k,\mathbb{R}) \tag{7}$$
$$n \mapsto \quad c(n) \quad \mapsto \quad \mathrm{Ad}(n). \tag{8}$$

Then the image of L in (8) is precisely the subgroup $\mathrm{GL}(k,\mathbb{Z})$, which is discrete in $\mathrm{GL}(k,\mathbb{R})$. Therefore, since N_0 is connected, the image of N_0 under the above map has to be the identity in $\mathrm{GL}(k,\mathbb{R})$. In other words, N_0 acts trivially on T by conjugation.

As a consequence, for any one-parameter group $\alpha : \mathbb{R} \to N_0$ the subgroup

$$\alpha(\mathbb{R}) \cdot T := \{\alpha(a)t : a \in \mathbb{R}, t \in T\} \subseteq G \tag{9}$$

is (as a continuous image of connected spaces) connected and abelian: $\forall a, b \in \mathbb{R}$ and $\forall t_1, t_2 \in T$ we have

$$\begin{aligned}
\alpha(a)t_1\alpha(b)t_2 &= \alpha(a+b)\underbrace{\alpha(b)^{-1}t_1\alpha(b)}_{=t_1}t_2 && (10)\\
&= \alpha(a+b)t_1t_2 && (11)\\
&= \alpha(b+a)t_2t_1 && (12)\\
&= \alpha(b+a)\underbrace{\alpha(a)^{-1}t_2\alpha(a)}_{=t_2}t_1 && (13)\\
&= \alpha(b)t_2\alpha(a)t_1 && (14)
\end{aligned}$$

where we used the fact that N_0 acts trivially (by conjugation) on T. Thus by maximality of T we have $\alpha(\mathbb{R}) \cdot T = T$ and in particular $\alpha(\mathbb{R}) \subseteq T$.
Since $\exp : LG \to G$ is a local diffeomorphism around 0 (since the derivative at 0 is the identity), for any g in a neighborhood of $e \in G$ there is a one-parameter subgroup containing g. Such a subgroup is given by $t \mapsto \exp(t\log(g))$, where log denotes the local inverse of exp. Hence the one-parameter subgroups cover an open neighborhood of e in N_0, which thus is also contained in T. Thus, since N_0 is connected, by Lemma 1.7 this open neighborhood generates N_0 and hence $N_0 \subseteq T$, which concludes the proof. □

Lemma 1.7. *Let G be a topological group, let $G_0 \subseteq G$ be its connected component, and let U be an open neighborhood of $e \in G$ contained in G_0, and let $\langle U\rangle$ denote the subgroup generated by U. Then $\langle U\rangle = G_0$.*

Proof. We want to show that the subgroup $\langle U\rangle$, which is generated by U is non-empty, open, and closed. Assume without loss of generality $U^{-1} \subseteq U$ (otherwise pass to $U \cap U^{-1}$).

1. Non-empty: Since $e \in U \subseteq \langle U\rangle$ the subgroup is non-empty.
2. Open: For any $g \in \langle U\rangle$ we have $g \cdot U \subseteq \langle U\rangle$.
3. Closed: If $g \notin \langle U\rangle$, then $g \cdot U \cap \langle U\rangle = \emptyset$, as otherwise $gu = v$ for some $u \in U, v \in \langle U\rangle$ and thus $g = vu^{-1} \in \langle U\rangle$. A contradiction. Hence the complement of $\langle U\rangle$ is also open.

□

2. Conjugates of Maximal Tori in G

The main theorem of this talk will be the following:

Theorem 2.1. *Let T and T' be two maximal tori in G. Then*

(1) the conjugate of T is again a maximal torus,

(2) T and T' are conjugate; i.e. there exists a $g \in G$ s.t. $T' = gTg^{-1}$,

(3) for any $g \in G$ there exists a maximal torus T in G s.t. $g \in T$, and

(4) the Weyl group is unique up to conjugation.

Its proof relies on the mapping degree of conjugations of the torus.

Theorem 2.2 (Mapping Degree; [BtD, I. (5.19)])**.** *Let M, N be compact, connected, oriented, n-dimensional manifolds and let $f : M \to N$ be a (homotopy class of a) differentiable function. Then there is an integer* $\deg(f)$ *such that for any $\alpha \in \Omega^n(N)$ we have*

$$\int_M f^*\alpha = \deg(f) \cdot \int_N \alpha. \tag{15}$$

If $q \in N$ such that $f^{-1}(q)$ consists of $k + l$ points $p_1, \ldots, p_{k+l}$ such that q is a regular value of f (i.e. that Df is bijective at every p_i), preserves orientation at $p_1, \ldots, p_k$ and reverses orientation at $p_{k+1}, \ldots, p_{k+1}$, then $\deg(f) = k - l$. *In particular, if f is not surjective, then there exists a $q \in N$ such that $f^{-1}(q) = \emptyset$ and thus* $\deg(f) = 0$.

In particular, it is a consequence of the following important lemma, the proof of which will be split up into several parts.

Lemma 2.3. *Let T be a maximal torus in G. Then the map*

$$q : G/T \times T \to G; \quad (g, t) \mapsto gtg^{-1} \tag{16}$$

has mapping degree $\deg(q) = |W|$, *where $|W|$ is the order of the Weyl group W associated to T. In particular, since $|W| > 0$, q is surjective.*

The proof of Lemma 2.3 is rather lengthy. Let us therefore first prove Theorem 2.1 from Lemma 2.3 and then turn to a proof of Lemma 2.3.

Proof of Thm. 2.1 from Lem. 2.3. (1) Let $g \in G$. Since $x \mapsto gxg^{-1}$ is a diffeomorphism $G \to G$ and T is compact and connected, so is gTg^{-1}. Commutativity and maximality of gTg^{-1} follow immediately from the commutativity and maximality of T.

(2) Let T and T' be two maximal tori in G. Let t' be a generator[2] of T'. By Lemma 2.3 there is a $g \in G$ such that $t' \in gTg^{-1}$ and hence, since t' generates T' we have $T' \subseteq gTg^{-1}$ and thus, by maximality of T', $T' = gTg^{-1}$.

[2] Recall that $t \in T$ is called a **generator of** T if the generated subgroup $\{t^k : k \in \mathbb{Z}\}$ is dense in T.

(3) Since q is surjective, every $g \in G$ is contained in gTg^{-1} for some $g \in G$, which, by (1) is again a maximal torus.

(4) Let N and N' be the normalizers of T and T', and let W and W' be the resulting Weyl groups, respectively. Let $g \in G$ be such that $T' = gTg^{-1}$, the existence of which is proven in (2). Then for any $n \in N, t' = gtg^{-1} \in T$ we have

$$(gng^{-1})t'(gng^{-1})^{-1} = gn\underbrace{g^{-1}t'g}_{=t}n^{-1}g^{-1} = g(ntn^{-1})g^{-1} \in gTg^{-1} = T' \tag{17}$$

Hence, $gng^{-1} \in N'$, and, by assumption $gTg^{-1} \subseteq T'$. Hence conjugation by g decends form N to $N/T = W$ and thus provides an isomorphism (with inverse being conjugation by g^{-1}). Thus the Weyl group is unique up to isomorphism (given by conjugation).

□

2.1. Some Observations

We will want to use the second part of Theorem 2.2 to compute the mapping degree of q from its fibre and conclude that q is surjective. However, for this we do not only need to understand the cardinality of the fibre of q (easy), but also the effect of q on the orientation (hard). For the latter we need orientations on $G/T \times T$ and on G that facilitate computation (e.g. are left-invariant, etc.; see Observations 2.4 and 2.6), and identifications (see. Observation 2.5) that allow the tangent map q_* to be understood as an endomorphism (not just a linear map), which then allows a computation of the determinant in the classical sense.

Before we turn to a proof of Lemma 2.3, let us first make some observations about the map

$$q : G/T \times T \to G; \quad (g, t) \mapsto gtg^{-1} \tag{18}$$

from Lemma 2.3 and the involved spaces.

Observation 2.4. *The map q is a smooth map between orientable compact manifolds of equal dimension. Note also that while T is generally not normal in G, and thus the space G/T does not carry a natural group structure, there still are the following natural left-actions:*

$$G \times T \curvearrowright G/T \times T, \quad G \curvearrowright G. \tag{19}$$

Proof. As a quotient of a compact Lie group G by a compact and connected subgroup T the orbit space G/T is a compact and orientable[3] manifold of dimension[4] $\dim(G) - \dim(T)$. The Lie groups T and G are orientable because they are parallelizable and compact and connected by assumption. □

[3]The orbit space is orientable since T is connected - see Prof. Dr. Leeb's Lecture Notes "General Facts about Lie Groups", end of page 7.

[4]See [Lee, Thm. 21.10], the "Quotient Manifold Theorem".

Observation 2.5. *Let $\langle\cdot,\cdot\rangle$ be an Ad_G-invariant inner product*[5] *on LG, let $LT \subseteq LG$ denote the Lie algebra of T, and let $LT^\perp \subseteq LG$ denote its orthogonal complement (w.r.t. $\langle\cdot,\cdot\rangle$) in LG. Then the splitting*

$$LG = LT^\perp \oplus LT \cong L(G/T) \oplus LT \tag{20}$$

is Ad_T invariant. As a consequence of the invariance, there is an induced action

$$Ad_{G/T} : T \to Aut(L(G/T)). \tag{21}$$

For the rest of the talk we will make the identification $LT^\perp \oplus LT \cong L(G/T) \oplus LT$.

Proof. 1) To see that LT is Ad_T-invariant, let $t \in T$ and $X \in LT$. Then for any $s \in \mathbb{R}$, using the fact that $c(t) \circ \exp = \exp \circ \mathrm{Ad}_t$, we have

$$\underbrace{\exp(sX)}_{\in T} = t \exp(sX) t^{-1} = c(t)\exp(sX) = \exp(s \,\mathrm{Ad}_t X). \tag{22}$$

Differentiating both sides of the resulting equation in s and evaluating at $s = 0$ gives $X = \mathrm{Ad}_t X$.

2) To see that $LT^\perp$ is invariant, let $t \in T$ and $X \in LT^\perp$; i.e. let $\langle X, Y\rangle = 0$ for any $Y \in LT$. Hence, by Ad_T invariance of LT and the inner product

$$\langle \mathrm{Ad}_t X, \mathrm{Ad}_t Y\rangle = \langle X, Y\rangle = 0. \tag{23}$$

Since $\mathrm{Ad}_t \in \mathrm{Aut}(LT)$, this implies that $\langle \mathrm{Ad}_t X, Z\rangle = 0$ for every $Z \in LT$ and hence $\mathrm{Ad}_t X \in LT^\perp$.

3) To see the second equality in (20) let $g \in G$ be arbitrary and let $\pi : G \to G/T$ denote the projection map. On the one hand, consider a smooth curve $\gamma : (-1, 1) \to gT \subseteq G$ such that $\gamma(0) = g$. Then

$$(\pi_*)_*(\gamma'(0)) = (\underbrace{\pi \circ \gamma}_{=[g]\in G/T})'(0) = 0 \tag{24}$$

Hence $T_g gT \subseteq \ker(\pi_*)_*$. On the other hand, by [Lee, Thm. 21.10], the "Quotient Manifold Theorem", π is a submersion. Thus $(\pi_*)_g : T_g G \to T_{gT}(G/T)$ is surjective and hence $\dim \ker \mathrm{d}\pi_g \leq \dim T$. Thus $T_g gT = \ker \mathrm{d}\pi_g$ and hence $T_{gT}(G/T) \cong T_g G/T_g gT$. □

Observation 2.6. *There are unique (up to choice of sign) invariant (under the actions in* (19)*) volume forms*

$\mathrm{d}g$	*on*	G	*invariant under action of*	G
$\mathrm{d}(gT)$	*on*	G/T	*invariant under action of*	G
$\mathrm{d}t$	*on*	T	*invariant under action of*	T

Each of them may be constructed by choosing a top-dimensional alternating form at the (image of) the identity and then defining the form at a point by pulling back through left-multiplication of the action.

[5] See Proposition A.2

In particular, $\pi^* \,\mathrm{d}(gT) \in \Omega^{n-k}(G)$ *and* $pr_2^*((\mathrm{d}t)_e) \in Alt^k(LG)$, *where* $pr_2 : LG = L(G/T) \oplus LT \to LT$. *Now, (via pullback along left-multiplication) the alternating k-form* $pr_2^*((\mathrm{d}t)_e)$ *determines a left-invariant k-form* $\mathrm{d}\tau \in \Omega^k(G)$ *such that* $\mathrm{d}\tau|_T = \mathrm{d}t$ *and such that* $\pi^* \,\mathrm{d}(gT) \wedge \mathrm{d}\tau$ *is a left-invariant volume form on G, since both parts of the wedge are left-invariant. Hence, we may choose the signs of the forms* $\mathrm{d}g$, $\mathrm{d}(gT)$, *and* $\mathrm{d}t$ *such that* $\pi^* \,\mathrm{d}(gT) \wedge \mathrm{d}\tau = c \cdot \mathrm{d}g$ *for some* $c > 0$. *One can show*[6] *that* $c = 1$, *but this is not important for our concerns.*

2.2. The determinant of the conjugation map q

Recall that for a smooth map $\varphi : M \to N$ between n-dimensional smooth manifolds, the pushforward induces a map on vector fields $\varphi_* : \Gamma(TM) \to \Gamma(TN)$, which in turn, via pullback, induces a linear map on n-forms: $\varphi^* : \Omega^n N \to \Omega^n M$. Since for each p the spaces $\mathrm{Alt}^n(T_pM)$ and $\mathrm{Alt}^n(T_{\varphi(p)}N)$ are 1-dimensional, this map can be be given, after choice of section α and β, by a real valued function $\det(\varphi) : M \to \mathbb{R}$ which then is defined by

$$\varphi^* \alpha = \det(\varphi)\beta. \tag{25}$$

If $M = N$ there is a canonical choice: $\alpha = \beta$; which then makes $\det(\varphi)$ independent of the choice of α. However, if $M \neq N$, a choice has to be made.

Observation 2.6 provides us with two reasonable volume forms on G and $G/T \times T$, respectively:

$$\mathrm{d}g = \pi^* \,\mathrm{d}(gT) \wedge \mathrm{d}\tau \in \Omega^n(G), \quad \mathrm{d}\tau|_T = \mathrm{d}t \in \Omega^k(T), \tag{26}$$

$$\alpha = \mathrm{pr}_1^* \,\mathrm{d}(gT) \wedge \mathrm{pr}_2^* \,\mathrm{d}t \in \Omega^n(G/T \times T). \tag{27}$$

which are invariant under the actions in (19). With the identification (20) we further have

$$\alpha_{(eT,e)} = \mathrm{d}g_e. \tag{28}$$

Definition 2.7. The **determinant** $\det(q) : G/T \times T \to \mathbb{R}$ of the conjugation map q is defined by the equation

$$q^* \,\mathrm{d}g = \det(q) \cdot \alpha. \tag{29}$$

Proposition 2.8. *For every* $(gT, t) \in G/T \times T$ *the determinant of the conjugation map* $q : G/T \times T \to G$ *at* (gT, t) *is given by*

$$\det(q)(gT, t) = \det(Ad_{G/T}(t^{-1}) - id_{L(G/T)}), \tag{30}$$

where $id_{L(G/T)}$ *is the identity map on* $L(G/T)$. *The determinant is to be understood as that of an endomorphism of* $L(G/T) \cong LT^{\perp}$.

Proof. In this proof, let us write $[g] := gT$ for equivalence classes in G/T, let $\ell.$ denote the left-action in (19) and let $([g], t) \in G/T \times T$ be fixed throughout the proof. We want to use the invariance of the involved forms to reduce the

[6] See [BtD, p. 160, 161].

computation of the determinant at $([g], t) \in G/T \times T$ to a computation at (eT, e). For this, consider the function

$$\varphi : G/T \times T \xrightarrow{\ell_{(g,t)}} G/T \times T \xrightarrow{q} G \xrightarrow{\ell_{gt^{-1}g^{-1}}} G. \tag{31}$$

Using invariance of the volume forms and the definition of the determinant of q we have

$$\begin{aligned}
\varphi^* \,\mathrm{d}g &= \ell^*_{(g,t)}(q^*(\ell^*_{gt^{-1}g^{-1}}\,\mathrm{d}g))) && (32)\\
&= \ell^*_{(g,t)}(q^*(\mathrm{d}g)) && (33)\\
&= \ell^*_{(g,t)}(\det(q)\cdot\alpha) && (34)\\
&= \det(q)\cdot(\ell^*_{(g,t)}\alpha) && (35)\\
&= \det(q)\cdot\alpha, && (36)
\end{aligned}$$

and $\varphi(eT, e) = (gt^{-1}g^{-1})gtg^{-1} = e$. Hence

$$(\varphi^*\,\mathrm{d}g)_{([e],e)} = \det(q)(g,t)\cdot\alpha_{([e],e)}. \tag{37}$$

and thus the computation of $\det(q)([g], t)$ is reduced to that of $(\varphi^*\,\mathrm{d}g)_{([e],e)}$. By (28) this amounts to computing the transformation of a degree n alternating tensor under pullback which can be done by computing the differential of φ at $([e], e)$ as an endomorphism

$$L(G/T)\oplus LT \to L(G/T)\oplus LT. \tag{38}$$

For any $([h], s) \in G/T \times T$ we may rewrite the application of φ as

$$\begin{aligned}
\varphi([h], s) &= \ell_{gt^{-1}g^{-1}}(q(\ell_{([g],t)}([h], s)) && (39)\\
&= \ell_{gt^{-1}g^{-1}}(q([gh], ts)) && (40)\\
&= \ell_{gt^{-1}g^{-1}}(([gh])ts[(gh)^{-1}]) && (41)\\
&= (gt^{-1}g^{-1})(([gh])ts[(gh)^{-1}]) && (42)\\
&= gt^{-1}[h]ts[h^{-1}]g^{-1} && (43)\\
&= c_g(c_{t^{-1}}([h])s[h^{-1}]). && (44)
\end{aligned}$$

Thus, using the chain rule and the product rule, the differential at $([e], e)$ is given by

$$(X, Y) \mapsto \mathrm{Ad}(g)\circ(\mathrm{Ad}_{G/T}(t^{-1})X + Y - X), \tag{45}$$

where $\mathrm{Ad}_{G/T}$ denotes the induced action in (21). Since the inner product on LG is Ad_G-invariant i.e. $\mathrm{Ad}(g)$ is orthogonal w.r.t. this inner product, the determinant of $\mathrm{Ad}(g)$ is ± 1. Since $\mathrm{Ad}(e) = \mathrm{id}_{LG}$ and G is connected, we have $\mathrm{Ad}(g) = 1$. Using the identification (20) and the Ad_T-invariance of the splitting in Observation 2.5, this gives an endomorphism in block form, whose determinant is thus

$$\det\begin{pmatrix}\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)} & 0\\ 0 & \mathrm{id}_{LT}\end{pmatrix} = \det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)}) \tag{46}$$

This concludes the proof. □

Lemma 2.9. *Let $t \in T$ be a topological generator. Then*

1) $q^{-1}(t)$ consists of $|W|$ many points and

2) $\det(q)(gT, s) > 0$ for any $(gT, s) \in q^{-1}(t)$

Proof. 1) Let $N(T)$ denote the normalizer of T in G and assume that $t \in T$ is a topological generator of T. Then for a fixed $gT \in G/T$

$$\exists s \in T : q(gT, s) = gsg^{-1} = t \tag{47}$$
$$\Leftrightarrow \exists s \in T : g^{-1}tg = s \in T \tag{48}$$
$$\Leftrightarrow g^{-1}Tg \subseteq T \tag{49}$$
$$\Leftrightarrow g \in N(T). \tag{50}$$

Therefore

$$q^{-1}(t) = \{(gT, g^{-1}tg) \in G/T \times T : g \in N(T)\} \tag{51}$$

Now, note that if two elements $(gT, g^{-1}tg)$, $(hT, h^{-1}th)$ in $q^{-1}(t)$ are equal if and only if $h^{-1}g \in T$. Thus $q^{-1}(t)$ is in bijection to $W = N(T)/T$ which gives the result.
2) Recall from Proposition 2.8 that $\det(q)$ is given by the determinant of an endomorphism of $L(G/T)$. We want to show that this endomorphism has no real eigenvalues. If that is the case, as an endomorphism of a real vector space, the eigenvalues come in complex conjugated pairs and thus the determinant (as a product of eigenvalues) is non-negative. Moreover, since this implies that 0 cannot be an eigenvalue, this implies that the determinant is strictly positive.
Firstly, if $\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)}$ had a real eigenvalue, then so would $\mathrm{Ad}_{G/T}(t^{-1})$ (since $-\mathrm{id}_{L(G/T)}$ just shifts the spectrum of $\mathrm{Ad}_{G/T}(t^{-1})$ by -1). Since, w.r.t. the Ad_G invariant inner product, $\mathrm{Ad}_{G/T}(t^{-1})$ is an orthogonal transformation, that eigenvalue would have to be ± 1. In that case, since $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$ that would imply that $\mathrm{Ad}_{G/T}(t^{-2})$ had eigenvalue 1. We show that this is a contradiction:
Assume there exists a non-zero $X \in L(G/T) \subseteq LG$ such that $\mathrm{Ad}_{G/T}(t^{-2})X = X$ and let $s \in \mathbb{R}$ be arbitrary. Then by linearity of the adjoint representation and naturality of the exponential map $\exp : G \to LG$ we have

$$c(t^{-2}) \exp(sX) = \exp(\mathrm{Ad}_{G/T}(t^{-2})sX) = \exp(sX), \tag{52}$$

and hence

$$c(t^{-2k}) \exp(sX) = \exp(sX), \quad k \in \mathbb{Z}\,. \tag{53}$$

By Kronecker's theorem A.3, t^{-2} is also a topological generator and hence

$$c(t') \exp(sX) = \exp(sX), \quad \forall t' \in T. \tag{54}$$

Thus the one parameter subgroup $H := \{\exp(sX) | s \in \mathbb{R}\}$ is left pointwise invariant by conjugation of T, i.e. every element in H commutes with every element in T. Thus $H \cdot T$ is abelian, compact and connected. Therefore $H \cdot T \subseteq T$ and hence $H \subseteq T$. Therefore $X \in LT \cap L(G/T) = \{0\}$. A contradiction. □

The following is a nice consequence of the proof above:

Observation 2.10. *If t topologically generates T, then $Ad_{G/T}(t)$ operates on $L(G/T)$ and has no real eigenvalues. Hence the dimension of G/T is even.*

Finally, let us complete the proof of Lemma 2.3.

Proof of Lemma 2.3. By (1) of Lemma 2.9 $q^{-1}(t)$ consists precisely of $|W|$ many points. By (2) of Lemma 2.9, q is orientation preserving at each of these points. Hence as a consequence of the second part of Theorem 2.2

$$\deg(q) = |W| > 0, \tag{55}$$

and thus by the last part of Theorem 2.2 q is surjective. □

Proposition 2.11 (Weyl's Integration Formula)**.** *Let $f : G \to \mathbb{R}$ be continuous. Then*

$$|W| \cdot \int_G f(g)\,\mathrm{d}g = \int_T \left[\det(id_{L(G/T)} - Ad_{G/T}(t^{-1})) \int_G f(gtg^{-1})\,\mathrm{d}g\right] \mathrm{d}t. \tag{56}$$

Proof. Via Lemma 2.3 and the definition of the mapping degree, the left hand side becomes

$$|W| \cdot \int_G f(g)\,\mathrm{d}g = \deg(q) \cdot \int_G f(g)\,\mathrm{d}g = \int_{G/T\times T} q^*(f\,\mathrm{d}g) = \int_{G/T\times T} (f\circ q) q^*\,\mathrm{d}g. \tag{57}$$

By (29) and (27) this gives

$$\int_{G/T\times T} (f\circ q) \underbrace{q^*\,\mathrm{d}g}_{=\det(q)\alpha} = \int_{G/T\times T} (f\circ q)\det(q)(\mathrm{pr}_1^*(\mathrm{d}gT)\wedge \mathrm{pr}_2^*\,\mathrm{d}t). \tag{58}$$

By Fubini's theorem and Proposition 2.8 we obtain

$$= \int_T \left(\int_{G/T} (f\circ q)\det(q)\,\mathrm{d}gT\right)\mathrm{d}t \tag{59}$$

$$= \int_T \left(\det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{id}_{L(G/T)}) \int_{G/T} (f\circ q)\,\mathrm{d}gT\right)\mathrm{d}t. \tag{60}$$

Finally, writing out q as a function on G instead of G/T this yields

$$= \int_T \left[\det(\mathrm{id}_{L(G/T)} - \mathrm{Ad}_{G/T}(t^{-1})) \int_G f(gtg^{-1})\,\mathrm{d}g\right]\mathrm{d}t. \tag{61}$$

In the last step we also used the fact that the dimension of $L(G/T)$ is even (noted in Observation 2.10), to switch sign inside the determinant. □

An interpretation of the formula: For a fixed t in the maximal torus T, define $f_t(g) = f(gtg^{-1})$. Note that f_t is constant on cosets of T and f factors into $f = f_t \circ \pi$. We may thus express the integral of f on G by first holding t fixed, integrating over the orbit gT, then weighing the result by the factor $\det(\mathrm{id}_{L(G/T)} - \mathrm{Ad}_{G/T}(t^{-1}))$ and integrating the result over T. In this sense, if we normalize $\mathrm{vol}(G) = \mathrm{vol}(G/T) = 1$, then $\det(\mathrm{id}_{L(G/T)} - \mathrm{Ad}_{G/T}(t^{-1}))$ can be interpreted as the volume of the conjugacy class of t.

A. Some Further Propositions

Lemma A.1. *Let N be a connected C^∞-manifold and let M be a compact C^∞-submanifold with inclusion $\iota : M \hookrightarrow N$. Then $dim(M) < dim(N)$, unless M and N are diffeomorphic.*

Proof. Recall that if ι is an immersion, then for any $p \in M$ the map $(D\iota)_p : T_pM \to T_{\iota(p)}N$ is injective and hence

$$\dim(M) = \dim(T_pM) \leq \dim(T_{\iota(p)N} = \dim(N). \tag{62}$$

To see that $\dim(M)$ has to be *strictly* smaller than $\dim(N)$ assume $\dim(M) = \dim(N)$ and M and N are not diffeomorphic. Then ι_* is pointwise injective and $\dim(T_pM) = \dim(T_{\iota(p)}N)$ by assumption, we conclude that ι_* is also pointwise surjective and hence a submersion. In particular, ι_* is pointwise invertible and thus as local diffeomorphism. The map ι is thus an open map and hence $\iota(M) \subseteq N$ is open. Also, since M is compact, $\iota(M) \subseteq N$ is closed and therefore closed. Thus, since N is connected $\iota(M) = N$ and ι is surjective. Hence ι is a bijective local diffeomorphism and thus a global diffeomorphism. □

Proposition A.2. *Let $b : LG \times LG \to \mathbb{R}$ be an inner product on LG. Then*

$$LG \times LG \to \mathbb{R} \tag{63}$$

$$(X, Y) \mapsto \langle X, Y \rangle := \int_G b\left(Ad(g)X, Ad(g)Y\right) \mathrm{d}g, \tag{64}$$

is an Ad_G-invariant inner product, where dg denotes the bi-invariant Haar measure on G.

Proof. The right hand side is finite since the integrand is a continuous function on a compact topological space and thus bounded and since the Haar measure is finite.

Bi-linearity and positivity are immediate. Assume $X \in LG$ such that

$$0 = \langle X, X \rangle = \int_G b\left(\mathrm{Ad}(g)X, \mathrm{Ad}(g)X\right) \mathrm{d}g \tag{65}$$

Then $\forall g \in G : b\left(\mathrm{Ad}(g)X, \mathrm{Ad}(g)X\right) =$ and thus, since b is non-degenerate, $\mathrm{Ad}(g)X = 0$. Since $\mathrm{Ad}(g) \in \mathrm{Aut}(LG)$, this implies $X = 0$.

Ad_G invariance follows from the fact that $g \mapsto \mathrm{Ad}(g)$ is a homomorhism and from the right-invariance of dg. □

Theorem A.3 (Kronecker)**.** *Let* $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$*. Then* $\exp(v) \in T^n$ *is a topological generator if and only if* 1 *and* $v_1, \ldots, v_n$ *are linearly independent over* $\mathbb{Q}$*; i.e. for every* $q_0, q_1, \ldots, q_n \in \mathbb{Q}$

$$q_1 v_1 + \ldots + q_n v_n = q_0 \quad \Rightarrow \quad q_0 = q_1 = \ldots = q_n = 0. \tag{66}$$

Theorem A.4 ((Consequence of) Cartan's Theorem)**.** *Let* $A \subseteq G$ *be a closed subgroup of a Lie group* G*. Then* A *is an embedded Lie subgroup.*

References

[BtD] T. Bröckner, T. tom Dieck, *Representations of Compact Lie Groups*, Springer, 1985.

[Lee] J. M. Lee, *Introduction to Smooth Manifolds*, Springer, 2012.

Classification of Rank 1 Lie Groups

Gideon Chiusole

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Throughout these notes, let G be a connected, compact Lie group, let $N(H)$ denote the normalizer of a subgroup $H \subseteq G$ and let 0 denote the trivial group.

1 Definitions and Basic Results

Definition 1.1. The **rank** k of G is the dimension of its maximal torus.

The notion of rank above is well defined since all maximal tori in G are conjugate and thus have the same dimension. The dimension of a torus is its only invariant as a Lie group. While the Weyl group of a maximal torus is a more refined invariant, it seems reasonable that, even though only a crude invariant, the classification of Lie groups of low rank is possible. This is the case for $k = 0, 1$.

Theorem 1.2. *If G has rank $k = 0$, then G is trivial.*

Proof. Assume G is non-trivial, then since it is assumed to be connected, it must have dimension at least 1. Hence there is a non-zero element $X \in LG$. Let $s \mapsto \alpha(s) := \exp(sX)$ be the one-parameter subgroup tangent to X. Since exp is a local diffeomorphism, α is a non-trivial subgroup. Hence $\overline{\alpha(\mathbb{R})}$ is a torus in G. □

Theorem 1.3. *If G has rank $k = 1$, then either $G \cong U(1)$, $G \cong SO(3)$, or $G \cong SU(2)$. The latter two have dimension 3 and are non-commutative, while the first has dimension 1 and is abelian.*

The proof will proceed in a few steps.

2 The abelian case

Proof. Assume G is abelian, then since G is also compact and connected by assumption, it has to be a torus. Thus it is its own maximal torus. Hence $G \cong U(1)$. □

3 The non-abelian case

Note immediately that if G is not abelian, then $\dim G > 1$ since one-dimensional Lie algebras must be abelian.

Lemma 3.1. *Let α be a one-parameter subgroup of G which is not periodic. Then G contains a torus of dimension $k > 1$.*

Proof. The subgroup $\alpha(\mathbb{R})$ is a connected abelian subgroup. Then $\overline{\alpha(\mathbb{R})}$ is a compact, connected abelian group, and thus a torus. Assume the dimension of the torus was $k \leq 1$. If it were $k = 0$, then it were periodic, and if it were $k = 1$, then $\alpha(\mathbb{R})$ would be a connected subgroup of $\overline{\alpha(\mathbb{R})} \cong U(1)$ which is dense, of which the only example is $U(1)$ itself, which would show that α was periodic. A contradiction to the assumption. □

3.1 Identifying $G/T \cong \mathbb{S}^{n-1}$

From now on set $\dim G = n$ and equip G with a bi-invariant Riemannian metric. Such a metric exists, since G is assumed to be compact.

Let $X \in LG$ and let $s \mapsto \alpha(s) := \exp(sX)$ be the one-parameter subgroup tangent to X. Then by assumption and Lemma 3.1 α is periodic i.e. is has compact image and of dimension 1 and thus is a 1-torus. On the other hand, every 1-torus can be constructed that way. In other words, any one-parameter group α must close up, and the images of the various one-parameter groups are precisely the maximal tori of G.

Proposition 3.2. *The adjoint representation of G on LG restricts to the unit sphere (w.r.t. the bi-invariant Riemannian metric) in LG and the induced action*

$$\Psi : G \to O(n) = Aut(\mathbb{S}^{n-1}); \quad \Psi(g) = Ad(g) \tag{1}$$

is transitive. Here $\mathbb{S}^{n-1} \subseteq LG$ denotes the unit sphere (w.r.t. the inner product fixed on LG).

Proof. Fix $g \in G$. Since the Riemannian metric was assumed to be bi-invariant, $\mathrm{Ad}(g)$ is an isometry on LG. Hence the adjoint representation restricts to $\mathbb{S}^{n-1} \subseteq LG$.
Since X and $-X$ generate the same one-parameter group, the action descends to the projectivization $P(LG) \cong \mathbb{S}^{n-1}/\{\pm 1\}$; i.e. the pairs of antipodes in $\mathbb{S}^{n-1}$ and since all maximal tori are conjugate to each other, the action is transitive on $P(LG)$. Thus, the action on $\mathbb{S}^{n-1}$ has at most[1] 2 orbits. However, since these orbits are compact, admitting 2 orbits would imply that $\mathbb{S}^{n-1}$ is not connected, which is not the case since $n > 1$. Hence the action can only have one orbit and thus the action (1) is transitive. □

Proposition 3.3. *For a given $X \in \mathbb{S}^{n-1} \subseteq LG$ as above, the map*

$$\phi_X : G/T \to \mathbb{S}^{n-1} \subseteq LG; g \mapsto Ad(g)X \tag{2}$$

is a well-defined, G-equivariant bijection between compact homogeneous G-spaces.

Proof. *Well-defined*: Let $t \in T$. Then for any $a \in \mathbb{R}$, using the naturality of the exponential map,

$$\exp(a\,\mathrm{Ad}(t)X) = c(t)\underbrace{\exp(aX)}_{\in T} = \exp(aX). \tag{3}$$

[1]For example, one orbit could be the upper hemisphere, while the other is the lower hemisphere, where the action on the lower hemisphere is dictated by the one on the upper in order to respect the quotient.

Differentiating both sides in s and evaluating at $s = 0$ gives $\mathrm{Ad}(t)X = X$. Hence for any $t_1^{-1}t_2 \in T$ we have

$$\mathrm{Ad}(t_1^{-1})\,\mathrm{Ad}(t_2)X = \mathrm{Ad}(t_1^{-1}t_2)X = X. \tag{4}$$

That is, $\mathrm{Ad}(t_1)X = \mathrm{Ad}(t_2)X$.
Equivariant: Since G acts on G/T by left-multiplication and on $\mathbb{S}^{n-1} \subseteq LG$ via the adjoint representation, ϕ_X is equivariant.
G-spaces: By Proposition 3.2 the adjoint action on $\mathbb{S}^{n-1}$ has a single orbit and thus gives a G-space.
Injective: Assume $\mathrm{Ad}(g)X = \mathrm{Ad}(h)X$. Then $\mathrm{Ad}(h^{-1}g)X = X$. Then, for any $a \in \mathbb{R}$

$$c(h^{-1}g)\exp(aX) = \exp(a\,\mathrm{Ad}(h^{-1}g)X) = \exp(aX). \tag{5}$$

Since $T = \exp(\mathbb{R}\,X)$, this implies that $h^{-1}g$ lies in $Z(T)$, which coincides with T itself, since T is a maximal torus.
Surjective: Since the action Ψ is transitive, the map is surjective. □

Corollary 3.4. *The map ϕ_X is a diffeomorphism*

$$G/T \cong \mathbb{S}^{n-1}. \tag{6}$$

Proof. By the Equivariant Rank Theorem (Lee, 2012, Thm. 7.25), as a bijective G-equivariant map between G-spaces, ϕ_X is a diffeomorphism. □

3.2 Characterizing the Weyl Group

Proposition 3.5. *The image of the normalizer under the map ϕ_X is $\{X, -X\}$. Thus the degree of T in $N(T)$ is $[N(T) : T] = 2$, $W \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element of W acts on T by orientation reversal.*

Proof. Let $g \in N(T)$. Then, since $T = \exp(\mathbb{R}\,X)$ there exists a $a \in \mathbb{R}$ such that

$$\exp(\mathrm{Ad}_g X) = \underbrace{c(g)\exp(X)}_{\in T} = \exp(aX). \tag{7}$$

However, by (1) we conclude $a = \pm 1$. Thus, by Corollary 3.4 this means that T has precisely 2 cosets in N. Thus $[N(T) : T] = 2$ and $W \cong \mathbb{Z}/2\mathbb{Z}$. The action of the non-trivial element $n \in W$ on T is thus given by

$$c(n)\exp(aX) = \exp(a\,\mathrm{Ad}(n)X) = \exp(-aX), \quad a \in \mathbb{R}. \tag{8}$$

□

3.3 Deducing $\dim = 3$ (topological argument)

Since $T \cong \mathbb{S}^1$ and $G/T \cong \mathbb{S}^{n-1}$, their homotopy groups (at least in low degree) are very tractable. We will thus make a homotopy theoretic argument for the dimension of G. Recall that by (Bröcker und Tom Dieck, 2013, Thm. (4.3))

$$T \xrightarrow{i} G \xrightarrow{p} G/T \tag{9}$$

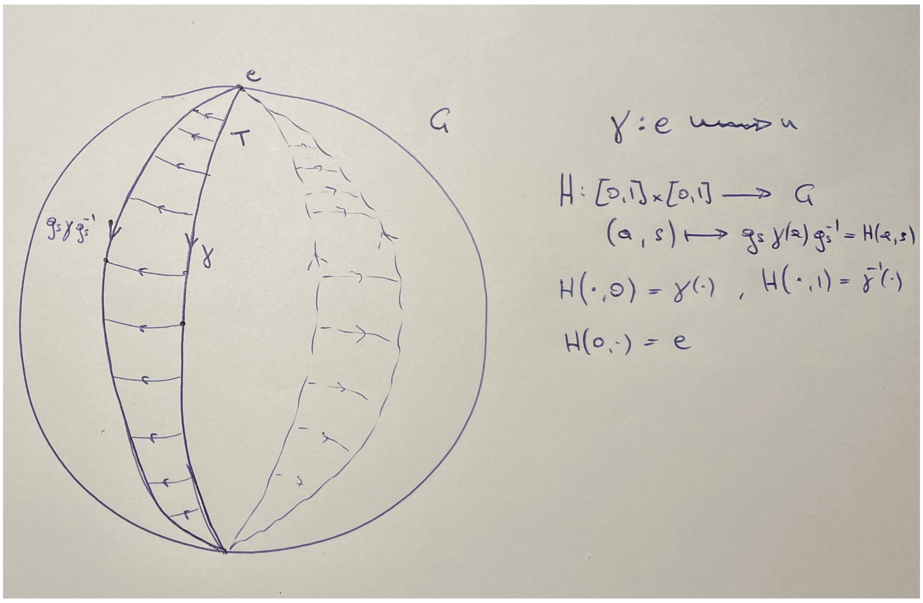

Figure 1: Illustration of $i_*([\gamma]) = i_*([\gamma])^{-1}$.

is a fibre bundle and therefore induces a long exact sequence in homotopy[2]

$$\ldots \to \pi_2(G/T) \xrightarrow{\delta} \underbrace{\pi_1(T)}_{\cong \mathbb{Z}} \xrightarrow{i_*} \pi_1(G) \xrightarrow{p_*} \pi_1(G/T) \to \ldots \tag{10}$$

where all homotopy groups are based at e and eT.

Proposition 3.6. *Let $[\gamma] \in \pi_1(T)$ be a generator of $\pi_1(T)$. Then $i_*([\gamma]) = i_*([\gamma])^{-1}$. In particular, $i_*([\gamma]) \in \pi_1(G)$ is of order 2.*

Proof. Since G is connected and thus, as a manifold, also path-connected, there is a path $g_\bullet : [0,1] \to G$ such that $g_0 = e$ and $g_1 = n$, where $n \in G$ is an element representing the non-trivial element in the Weyl group. By Proposition 3.5 we have $\mathrm{Ad}_n X = -X$. Let $[\gamma] \in \pi_1(T)$ be a generator as above. Then

$$H : [0,1] \times [0,1] \to G; \quad (a, s) \mapsto g_s \gamma(a) g_s^{-1} \tag{11}$$

is a homotopy in G from $H(\cdot, 0) = \gamma$ to $H(\cdot, 1) = \gamma^{-1}$, which fixes $H(0, \cdot) = e$. Hence γ and γ^{-1} are homotopic (relative to e) and thus represent the same element in $\pi_1(G)$. Hence $i_*([\gamma])^2 = i_*([\gamma]) i_*([\gamma])^{-1} = 1$.

□

As a result of Proposition 3.6 the image $i_*(\pi_1(T)) \subseteq \pi_1(G)$ is either isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or 0. Therefore, the kernel of i_* must be infinite, which, by the exactness of the sequence, implies that $\pi_2(G/T)$ is also infinite. Since $G/T \cong \mathbb{S}^{n-1}$ and the homotopy groups of spheres in low degree and dimension are

[2]See (Hatcher, 2005, Sec. 4.2).

known, this leaves only $n = 3$. It then follows furthermore that $\pi_1(G/T) = \pi_1(\mathbb{S}^{n-1}) = 0$, and then, again by exactness, that i_* is surjective. Therefore, $\pi_1(G)$ is either isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or 0.

3.4 Concluding via coverings

We now know that G must be of dimension $n = 3$ and that the $\pi_1(G)$ is either isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or 0. To conclude with a full classification we appeal to the theory of covering spaces.[3]

Recall that G acts on LG via the adjoint representation as isometries w.r.t. the bi-invariant inner product fixed in the beginning of the section; that is, Ψ, as defined in (1), takes values in the $\mathrm{Isom}(\mathbb{S}^2) = \mathrm{O}(3)$. More specifically, since G is connected, its image has to lie in the connected component containing e. We summarize:

$$\Psi : G \to \mathrm{Isom}(\mathbb{S}^2)_0 = \mathrm{SO}(3); g \mapsto \mathrm{Ad}(g). \tag{12}$$

Proposition 3.7. *The kernel of Ψ equals the center $Z(G)$ of G and is therefore discrete.*

Proof. On the one hand, let $g \in Z(G)$ and $X \in LG$. Then for any $a \in \mathbb{R}$

$$\exp(a\,\mathrm{Ad}(g)X) = c(g)\underbrace{\exp(aX)}_{\in T} = \exp(aX). \tag{13}$$

Differentiating both sides in a and evaluating at $a = 0$ shows that $g \in \ker \Psi$.

On the other hand, assume $g \in \ker \Psi$. Since G is compact and connected, exp is surjective[4] and hence for any $h \in G$ there is a $X_h \in LG$ such that $h = \exp(X_h)$ and thus

$$c(g)h = c(g)\exp(X_h) = \exp(\underbrace{\mathrm{Ad}(g)}_{=\mathrm{id}_{LG}} X_h) = \exp(X_h) = h, \quad \forall h \in G. \tag{14}$$

Thus $g \in Z(G)$. Hence, since the center of a compact Lie group equals the intersection of all its maximal tori[5], which are 1-dimensional in our case, the center is discrete. □

Therefore, SO(3) is the quotient of G by a finite subgroup (which acts properly). Hence $\dim(\mathrm{SO}(3)) = \dim G = 3$, Ψ is a submersion, and thus Ψ is a local diffeomorphism.

Thus G must be a covering space of SO(3). Luckily, the fundamental group of SO(3) is known to be $\mathbb{Z}/2\mathbb{Z}$, and thus, by standard covering space theory, there are (up to covering space isomorphism) precisely two covering spaces (corresponding to the subgroups of $\mathbb{Z}/2\mathbb{Z}$, which are $\mathbb{Z}/2\mathbb{Z}$ itself and 0). These are the trivial covering $\mathrm{SO}(3) \to \mathrm{SO}(3)$ and the 2:1-covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Thus G must be isomorphic to either SU(2) or SO(3).

[3] See (Hatcher, 2005, Sec. 1.3).
[4] See (Bröcker und Tom Dieck, 2013, Thm. (2.2))
[5] See (Bröcker und Tom Dieck, 2013, Thm. (2.3) (iii)).

References

[Bröcker und Tom Dieck 2013] Bröcker, Theodor ; Tom Dieck, Tammo: *Representations of compact Lie groups*. Bd. 98. Springer Science & Business Media, 2013

[Hatcher 2005] Hatcher, Allen: *Algebraic topology*. Cambridge University Press, 2005

[Lee 2012] Lee, John M.: *Smooth manifolds*. Springer, 2012