Gaussian Random Variables [Da Prato, 2006, Chap. 1 & 2]

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Note - Erratum from last week

Theorem

Let $Q \in L_1^+(H)$ and injective. Then there exists an ONB $(e_k)_{k\in\mathbb{N}}$ of H and a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^1$ s.t.

$$\forall k \in \mathbb{N} : Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0$$

in particular $\lambda_k \to 0$ as $k \to \infty$.

Otherwise we only get an orthonormal set and not an orthonormal basis.

From here on, Q is assumed to be non-degenerate.

New Formulation

Theorem

Let $Q \in L_1^+(H)$ and injective. Then there exists an ONB $(e_k)_{k\in\mathbb{N}}$ of H and a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^1$ s.t. $\forall k \in \mathbb{N} : \lambda_k \geq 0$ and

$$Qx = \sum_{k=1}^{\infty} \lambda_k \langle e_k, x \rangle, \quad x \in H$$

$$Q^{1/2}x = \sum_{k=1}^{\infty} \lambda_k^{1/2} \langle e_k, x \rangle, \quad x \in H$$

$$Q^{-1}x = \sum_{k=1}^{\infty} \lambda_k^{-1} \langle e_k, x \rangle, \quad x \in H$$

in particular, Q^{-1} is unbounded and only defined on $Q(H)$.

Gaussian Random Variables

Gaussian Random Variable

Definition

A random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (H, \mathcal{B}(H))$ is called Gaussian if it has Gaussian law i.e. if the measure $\mathbb{P} \circ X^{-1}$ is Gaussian.

Convergence of Gaussian RVs

Theorem

Let $(X_n)_{n\in\mathbb{N}}$ *be a sequence of* H*-valued* N_{a_n,Q_n}*-distributed RVs s.t.* $X_n \to X$ *in* L^2*. Then* X *has distribution* $N_{a,Q}$ *where*

$$\forall h, k \in H : \langle a_n, h\rangle \to \langle a, h\rangle, \quad \langle h, Q_n k\rangle \to \langle h, Qk\rangle \tag{1}$$

Translation and rescaling of Gaussian RVs

Theorem

Let $\mu = N_{a,Q}$ *be a Gaussian measure on* $(H, \mathcal{B}(H))$, $b \in H$, $T \in L(H, K)$. *Then the function* $h \mapsto Th + b$ *is Gaussian with distribution* N_{Ta+b, TQT^*}.

Translation and Rescaling of Gaussian RVs: Corollary

Corollary

Let $\mu = N_{0,Q}$ *be a Gaussian measure on* $(H, \mathcal{B}(H))$, $z_1, \ldots, z_n \in H$ *and* $T : H \to \mathbb{R}^n$ *defined by*

$$Tx = (\langle z_1, x\rangle, \ldots, \langle z_n, x\rangle), \quad x \in H. \tag{2}$$

Then T *is a Gaussian random variable with values in* $\mathbb{R}^n$ *and law* $N_{Q'}$ *where*

$$Q' = TQT^*, \quad \text{i.e. } Q'_{i,j} = \langle z_i, Qz_j\rangle, \quad i,j = 1, \ldots, n. \tag{3}$$

Non-degenerate Gaussian measures are full

Theorem

Let $\mu = N_{a,Q}$ *be a non-degenerate Gaussian measure on* $(H, \mathcal{B}(H))$*. Then the smallest open subset* $U \subseteq H$ *with* $\mu(U) = 1$ *is* H *itself.*

White Noise Mapping

White Noise Mapping

Definition

Let μ be a Gaussian measure on $(H, \mathcal{B}(H))$. The mapping $Q^{1/2}(H) \to L^2(H, \mu)$ defined by $z \mapsto W_z(x) := \langle Q^{-1/2}z, x\rangle$ is called the **white noise mapping**.

White Noise Mapping

Proposition

The white noise mapping is an isometry on $Q^{1/2}(H)$ *and can thus be uniquely extended to an isometry on* $\overline{Q^{1/2}(H)} = H$.

Proof.

For any $z_1, z_2 \in H$ we have

$$\begin{aligned}\int_H W_{z_1}(x) W_{z_2}(x) d\mu(x) &= \int_H \langle Q^{-1/2} z_1, x\rangle \langle Q^{-1/2} z_2, x\rangle d\mu(x) \\ &= \langle Q^{-1/2} z_1, Q Q^{-1/2} z_2\rangle = \langle z_1, z_2\rangle.\end{aligned}$$

Since Q is assumed to be injective, $\overline{Q^{1/2}(H)} = H$. □

Cameron-Martin Formula

Equivalence and Mutual Singularity of Measures

Let μ, ν be two measures on $(\Omega, \mathcal{F})$. Then μ, ν are called

- **equivalent** (in symbols $\mu \approx \nu$) if both $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ exist i.e. by Radon-Nikodym, if $\forall A \in \mathcal{F} : \mu(A) = 0$ if and only if $\nu(A) = 0$.
- **mutually singular** (in symbols $\mu \perp \nu$) $\exists A \in \mathcal{F} : \mu(A) = 1$ and $\nu(A) = 0$.

Equivalence and Mutual Singularity of Measures

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- **mutually singular** (in symbols $\mu \perp \nu$) $\exists A \in \mathcal{F} : \mu(A) = 1$ and $\nu(A) = 0$.

e.g. $\delta_0 \perp \delta_1, \delta_0 \perp \lambda^1, N_{0,1} \approx N_{1,1}, N_{0,1} \approx \lambda^1, \ldots$

But e.g. $\delta_2 + \text{Uni}[0,1]$ and λ^1 are neither mutually singular nor equivalent

Definition

Let μ, ν be two probability measures on a probability space $(\Omega, \mathcal{F})$ and let ζ be any measure on $(\Omega, \mathcal{F})$ s.t. $\frac{\mathrm{d}\mu}{\mathrm{d}\zeta}$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\zeta}$ exist. Then the **Hellinger integral** of μ and ν is defined as

$$H(\mu, \nu) = \int_\Omega \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\zeta}\frac{\mathrm{d}\nu}{\mathrm{d}\zeta}}\,\mathrm{d}\zeta \quad . \tag{4}$$

Hellinger Integral

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$H(\mu, \nu)$ is independent of the choice of ζ and Hölder's inequality we have

$$0 \le H(\mu, \nu) \le \left(\int_\Omega \frac{\mathrm{d}\mu}{\mathrm{d}\zeta}\mathrm{d}\zeta\right)^{\frac{1}{2}} \left(\int_\Omega \frac{\mathrm{d}\nu}{\mathrm{d}\zeta}\mathrm{d}\zeta\right)^{\frac{1}{2}} \le 1 \tag{5}$$

Example

Let $\mu := N_{0,\lambda}$, $\nu := N_{a,\lambda}$ be Gaussian measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $a \in \mathbb{R}$, $\lambda > 0$. Then

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \exp\left\{-\frac{a^2}{2\lambda} + \frac{ax}{\lambda}\right\}, \quad x \in \mathbb{R}. \tag{6}$$

and

$$H(\mu, \nu) = \exp\left\{-\frac{a^2}{8\lambda}\right\} \tag{7}$$

Raison d'être

Theorem

$\mu \perp \nu$ *if and only if* $H(\mu, \nu) = 0$.

Proof.

□

Raison d'être

If $\mu \approx \nu$ then $H(\mu, \nu) > 0$.

$$H(\mu, \nu) = \int_\Omega \sqrt{\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\zeta}\right)\frac{\mathrm{d}\nu}{\mathrm{d}\zeta}}\ \mathrm{d}\zeta \tag{8}$$

$$= \int_\Omega \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\nu}}\frac{\mathrm{d}\nu}{\mathrm{d}\zeta}\ \mathrm{d}\zeta \tag{9}$$

$$= \int_\Omega \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\nu}}\ \mathrm{d}\nu > 0 \tag{10}$$

The converse does not necessarily hold. However, for product measures, it does.

Mutual Singularity for Product Measures

Theorem (Kakutani)

Let $(\mu_k)_{k=1}^{\infty}$ *and* $(\nu_k)_{k=1}^{\infty}$ *be sequences of probability measures on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *with* $\mu := \bigtimes_{k=1}^{\infty} \mu_k$, $\nu := \bigtimes_{k=1}^{\infty} \nu_k$. *Then*

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k). \tag{11}$$

If $\mu_k \approx \nu_k$ *for every* $k \geq 0$ *and* $H(\mu, \nu) > 0$, *then* $\mu \approx \nu$ *and*

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \lim_{n \to \infty} \prod_{k=1}^{\infty} \left(\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu_k} \circ \pi_k \right) \in L^1(\mathbb{R}^{\infty}, \mu). \tag{12}$$

Proof.

See [Da Prato, 2006, Ex. 2.6, Thm 2.7]. □

Theorem (Cameron-Martin)

Let $\mu := N_{0,Q}$, $\nu := N_{a,Q}$ be Gaussian measures on $(H, \mathcal{B}(H))$ and $a \in H$. Then

(i) *if $a \notin Q^{1/2}(H)$, then $\mu \perp \nu$.*

(ii) *if $a \in Q^{1/2}(H)$, then $\mu \approx \nu$ and density is given by*

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \exp\left\{-\frac{1}{2}\|Q^{-1/2}a\|_H^2 + W_{Q^{-1/2}a}x\right\}, \quad x \in H. \tag{13}$$

Proof.

□

We have seen that translation along some $a \in H$ gives either an equivalent measure or a mutually singular. In fact, this holds true more generally.

Theorem (Feldman-Hajek)

Let $Q, R \in L_1^+(H)$ s.t. $QR = RQ$ and let $\mu := N_Q$, $\nu := N_R$. Then μ and ν are equivalent if and only if

$$\sum_{k=1}^{\infty} \frac{(\lambda_k - r_k)^2}{(\lambda_k + r_k)^2} < \infty \tag{14}$$

where λ_k and r_k denote the eigenvalues of Q and R, respectively. Otherwise they are mutually singular.

Proof.

□

Feldman-Hajek Theorem: Corollary

Corollary

Let $R = \alpha Q$ *with* $\alpha > 0$. *Then by the Feldman-Hajek Theorem, for* $\alpha \neq 1$ *we have*

$$N_{0,Q} \perp N_{0,R}. \tag{15}$$

Feldman-Hajek Theorem: Corollary

Corollary

Let $R = \alpha Q$ *with* $\alpha > 0$. *Then by the Feldman-Hajek Theorem, for* $\alpha \neq 1$ *we have*

$$N_{0,Q} \perp N_{0,R}. \tag{15}$$

Upshot: Measures on infinite-dimensional spaces have a strong tendency to be mutually singular. Recall here also that $\mu(Q^{1/2}(H)) = 0$.

Bibliography

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