Fundamental Group of $\mathbb{R}/\mathbb{Q}$

Abstract

In an algebraic-topological context, together with homology and cohomology, the fundamental group is among the first invariants of a topological space one should consider. In this exposition we want to compute the fundamental group of $\mathbb{R}/\mathbb{Q}$ and give a geometric interpretation of the space. We will firstly go over a strategy that seems promising but is ultimately misleading (which is essentially what makes the example interesting).

$[0, 1]$ modulo a finite set

To get a very first idea of the problem, note how the quotient of $\mathbb{R}$ behaves when only finitely many points are identified. A simple model of this is the space $[0, 1]/\{0 = t_0 < \ldots < t_n = 1\}$. It is homeomorphic to the n-fold bouquet of circles $X := \vee_{i=1}^{n} \mathbb{S}^1_i$. Via Seifert–Van Kampen or covering spaces, its fundamental group $\pi_1(X) = (a_1, \ldots, a_n)$ can be computed to be the free group on n generators.
Note here that this example makes just as much sense by considering $\mathbb{R}$ modulo $\{t_0 < \ldots < t_n\}$ instead. But in that case, the resulting space would not be homeomorphic, but only homotopy equivalent to the n-fold bouquet. But, of course, that does not change the fundamental group either.

$\mathbb{R}$ modulo the countably infinite, discrete set $\mathbb{Z}$

Now, consider the space $Y := \mathbb{R}/\mathbb{Z}$ i.e. the topological space with the underlying set of equivalence classes

$$\{x \in \mathbb{Z}\} \cup \{\{y\} : y \in \mathbb{R} \setminus \mathbb{Z}\} \tag{1}$$

equipped with the final topology. The natural quotient map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ maps every element of $\mathbb{Z}$ to a single point p and leaves the other points untouched. This space is homeomorphic (!) to an infinite bouquet of circles and thus its fundamental group is the free group on countably infinitely many generators.

$\mathbb{R}$ modulo the countably infinite, dense set $\mathbb{Q}$

Now take a guess about $\mathbb{R}/\mathbb{Q}$. The naive guess is a group on infinitely many generators which is either free, or has wildly non-free behaviour due to the density of $\mathbb{Q}$ in $\mathbb{R}$.
But, the fundamental group is trivial, since the space is contractible. To show this, denote $Z := \mathbb{R}/\mathbb{Q}$ and let $\xi := \pi(\mathbb{Q})$ be the image of $\mathbb{Q}$ under the natural quotient map $\pi : \mathbb{R} \to Z$. Firstly, note that $\{\xi\}$ is dense in Z. To see this, recall that a set $U \subseteq Z$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}$. So any open set containing x necessarily also contains ξ, otherwise $\pi^{-1}(U) \subseteq \mathbb{R}$ does not contain $\mathbb{Q}$ and cannot be open. Hence for any point $x \in Z$, the point ξ lies in every neighbourhood, and thus the constant sequence at ξ converges to any $x \in Z$.
Let $\phi : Z \to \{\xi\}$, and $\iota : \{\xi\} \hookrightarrow Z$ the inclusion, then we have $\phi \circ \iota = \mathrm{id}_{\{\xi\}}$ and want to show $\iota \circ \phi \sim \mathrm{id}_Z$ via the following homotopy:

$$H : [0, 1] \times Z \to Z \quad , \begin{cases} (t, x) \mapsto x, & t \in [0, \frac{1}{2}] \\ (t, x) \mapsto \xi, & t \in (\frac{1}{2}, 1] \end{cases} \tag{2}$$

We have $H(0, -) = \mathrm{id}_Z$ and $H(1, -) = \iota \circ \phi$ i.e. if we can show that H is continuous, then it is a homotopy between id_Z and $\iota \circ \phi$. To check continuity, let $(t_i, x_i)_{i \in I}$ be a net in $[0, 1] \times Z$, s.t. $t_i \to t$ and $x_i \to x$, then we want to show that $H(t_i, x_i) \to H(t, x)$. For this, consider the following cases:

- if $t \in [0, \frac{1}{2})$, then for large enough i_0 we have $t_i \in [0, \frac{1}{2})$ and thus $H(t_i, x_i) = x_i$ for every $i \geq i_0$ and thus $H(t_i, x_i) \to x = H(t, x)$
- if $t \in (\frac{1}{2}, 1]$, then for large enough i_0 we have $t_i \in (\frac{1}{2}, 1]$ and thus $H(t_i, x_i) = \xi$ for every $i \geq i_0$ and thus $H(t_i, x_i) \to \xi = H(t, x)$
- if $t = \frac{1}{2}$, then we want to show that $H(t_i, x_i) \to H(\frac{1}{2}, x) = x$. So let $U \subseteq Z$ be some neighborhood of x and choose $i_0 \in I$ s.t. $\forall i \geq i_0 : x_i \in U$. Such a i_0 exists since $x_i \to x$. Then we have that for every $i \geq i_0$ either $t_i \in [0, \frac{1}{2}]$ and thus $H(t_i, x_i) = x_i \in U$ or $t_i \in (\frac{1}{2}, 1]$ and thus $H(t_i, x_i) = \xi$, which is in U since $\{\xi\}$ was dense in Z.

Thus H is continuous, therefore a homotopy and thus ϕ is shown to be a deformation retract. So in particular, Z is contractible and thus, by the homotopy invariance of π_1 has trivial fundamental group.