Classification of Topological Surfaces

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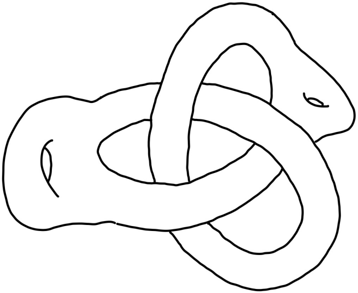

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History and A Priori Considerations

History [Gallier and Xu, 2012, Appx. D]

Studied since 1850s

A. F. Möbius 1863 early attempts at a classification theorem, introduced the family of genus g surfaces $\{\Sigma_g\}_{g\in\mathbb{N}}$

C. Jordan 1866 unaware of Möbius' work, version for surfaces with boundary
version of homeomorphism (correspondence of "infinitely small element"),
also considers genus, but does not recognize it as such

G. Cantor 1871 rigorous definition of $\mathbb{R}$

W. von Dyck 1888 incorporation of non-orientable surfaces, Klein bottle, first appearance of normal form

History [Gallier and Xu, 2012, Appx. D]

H. Poincare 1904 First considerations of 3-manifold, Poincare conjecture: "Every simply-connected, closed 3-manifold is homeomorphic to the 3-sphere"

M. Dehn, P. Heegaard 1907 first "rigorous" proof of classification

H. Weyl 1912 intrinsic definition of manifold

J. Alexander 1915 sketch of reduction to normal form as a quotient of $2n$-gon, first appearance of "labelling scheme"

History [Gallier and Xu, 2012, Appx. D]

H. Brahana 1920 first clear and justified proof of classification via "method of cutting"

W. Thurston 1982 proposed Thurston's geometrization conjecture, proof for Haken manifolds (Fields medal)

G. Perelman 2003 Sketch (and subsequent proof) of Thurston's geometrization conjecture (Fields medal 2006)

Difficulty varies drastically with Dimension

- $\dim = 0$ trivial: $\{*\}$
- $\dim = 1$ reasonable exercise: $\mathbb{S}^1$ or $\mathbb{R}$
- $\dim = 2$ $\sim$ 60 minute seminar talk: $\langle \Sigma, \mathbb{RP}^2 | P_3 \cong \mathbb{RP}^2 \# \Sigma \rangle$
- $\dim = 3$ resolved in the early 2000s: 8 prime geometries
- $\dim \geq 4$ open problem: ???, for any finitely presented group G there exists a smooth 4-manifold M with $\pi_1(M) \cong G$
  $\Rightarrow$ focus on simply-connected case

A Priori Considerations

What is a surface? When are two surfaces equal?

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Relevant candidates:

- $\mathbf{TopMan}$, the category of topological manifolds and continuous maps,
- $\mathbf{PLMan}$, the category of piecewise linear manifolds and piecewise linear maps, and
- $C^\infty\mathbf{Man}$, the category of smooth manifolds and smooth maps.

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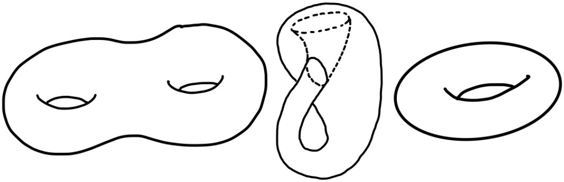
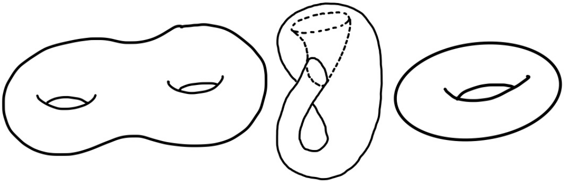
However, in $\dim \leq 3$ all equivalent.

First Simplification



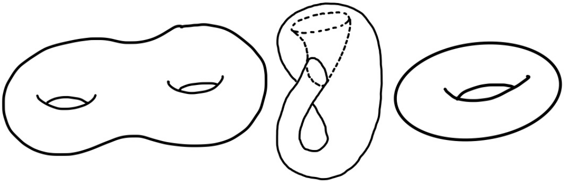
First Simplification



Lemma ([Lee, 2013, Prop. 1.11])
The connected components of a manifold M are again manifolds, and M is homeomorphic to their disjoint union:

$$M \cong \coprod_{N \in \pi_0(M)} N$$

First Simplification



Lemma ([Lee, 2013, Prop. 1.11])
The connected components of a manifold M are again manifolds, and M is homeomorphic to their disjoint union:

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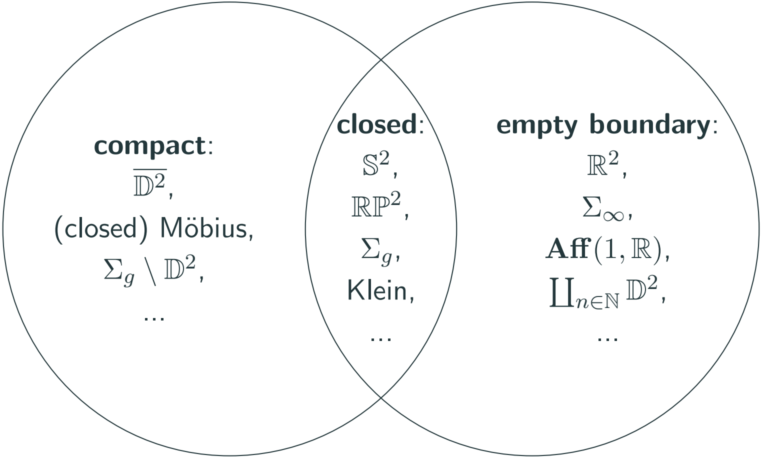
Hence reduce to: surface $\triangleq$ **connected** 2-manifold.

Further Subdivision



Classification of closed Surfaces

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Theorem

Let S be a closed surface. Then S is homeomorphic to exactly one member of the following families:

- *the* 2-**sphere** $\mathbb{S}^2$ *(*$=: \Sigma_0$*)*
- *the* **genus** g-**surface** $\Sigma_g := \underbrace{\mathbb{T}^2 \# \ldots \# \mathbb{T}^2}_{g \text{ times}}$
- *the* m-**fold projective plane** $P_m := \underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{m \text{ times}}$

Furthermore, $P_3 \cong \mathbb{RP}^2 \# \Sigma_1$ *is the only non-trivial relation.*

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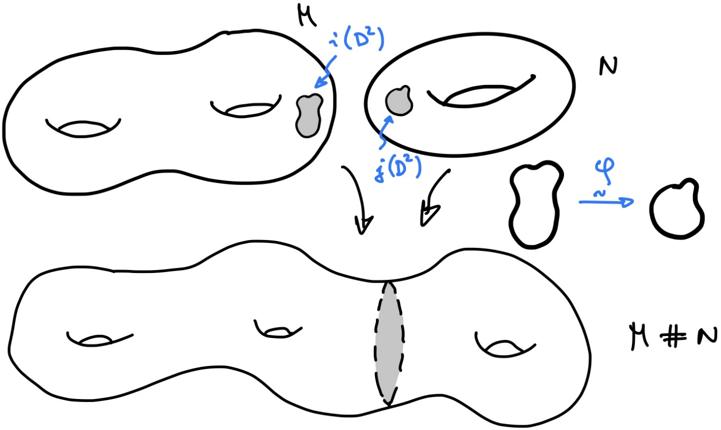
Furthermore, $P_3 \cong \mathbb{RP}^2 \# \Sigma_1$ *is the only non-trivial relation.*

In other words, there is a bijection

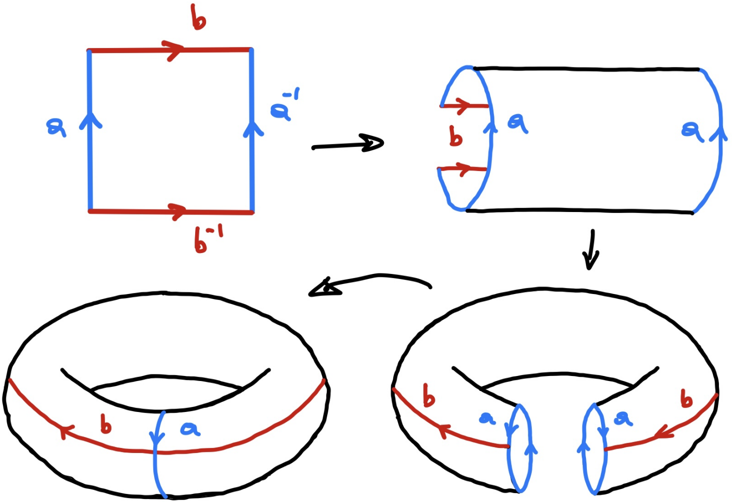
$$\{\text{closed surfaces}\}/\{\text{homeo}\} \to \langle \Sigma_1, \mathbb{RP}^2 | P_3 \cong \mathbb{RP}^2 \# \Sigma \rangle$$

Remark: Connected Sum

$$M\#N := M \setminus i(\mathbb{D}^2) \coprod_{\partial i(\mathbb{D}^2) \cong_{\varphi} \partial j(\mathbb{D}^2)} N \setminus j(\mathbb{D}^2)$$



First Observations: Torus as a Quotient of $[0, 1]^2$



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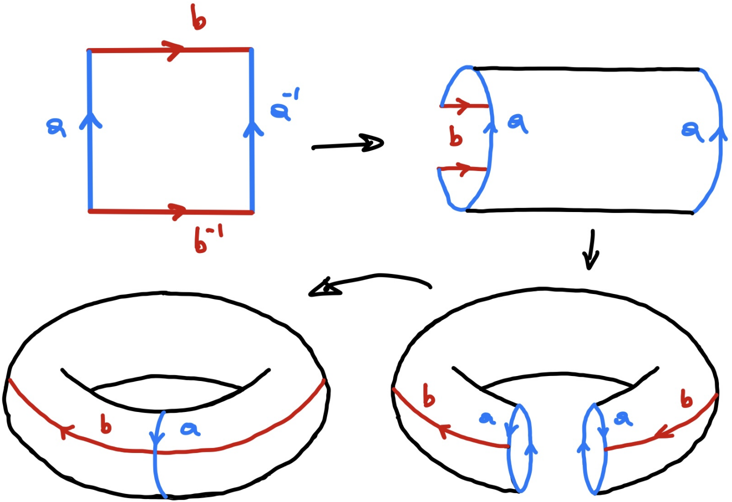
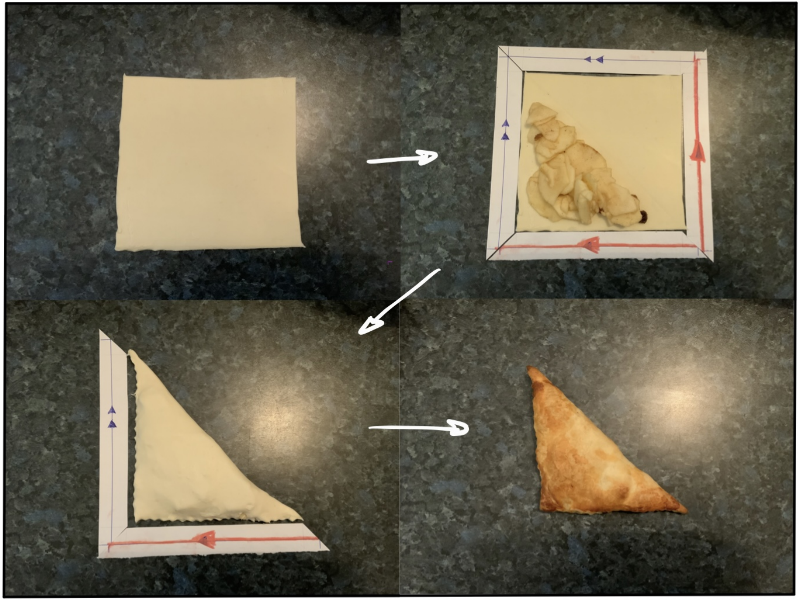


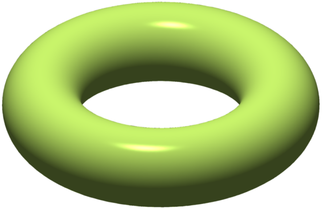
Realize $\mathbb{S}^2$ as a quotient of $[0,1]^2$?

How to make an Apfeltasche



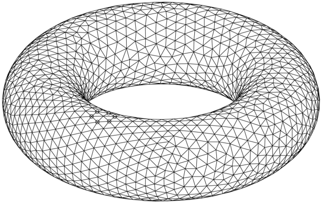
Triangulation[1]: Topology ⇒ Topology & Combinatorics

Surface



$\longrightarrow$

Triangulation



[1] https://en.wikipedia.org/wiki/Triangulation_(topology)#/media/File:Torus-triang.png, https://de.wikipedia.org/wiki/Volltorus#/media/Datei:Torus_illustration.png

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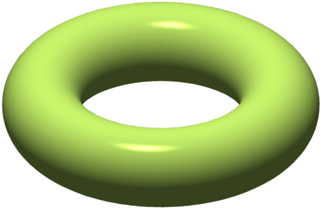
Surface | Triangulation



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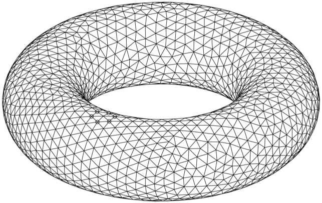


Theorem (Rado, Moise, Freedman, Casson)

Any topological manifold with $\dim \leq 3$ *has a triangulation.*

(pre-1930s this was implicitly assumed)

For dimensions > 3 *this is not true.*

Theorem (Cairns, Whitehead)

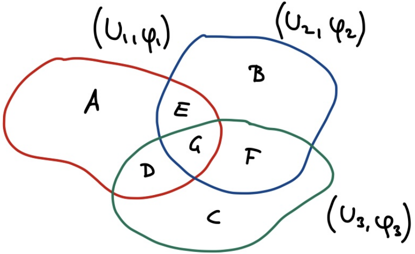
Any differentiable manifold has a triangulation.

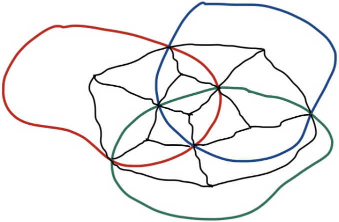
[1] https://en.wikipedia.org/wiki/Triangulation_(topology)#/media/File:Torus-triang.png, https://de.wikipedia.org/wiki/Volltorus#/media/Datei:Torus_illustration.png

Jordan Covering of finite Character

An atlas $\{(U_i, \varphi_i)\}$ is called a **Jordan covering of fin. char.** if for each i

- the chart-map φ_i can be extended to $\overline{U}_i$
- for each j $\partial U_i \cap \partial U_j$ consists of fin. many points or arches
- $U_i \cap U_j \neq \emptyset$ for at most fin. many j





Cutting open a triangulated Surface

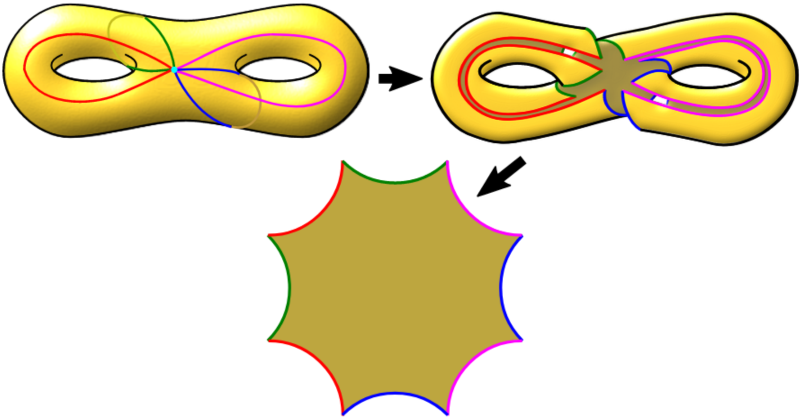


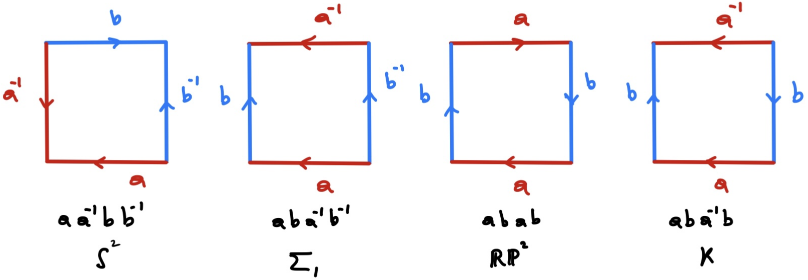
Figure 1: Cutting open triangulation [van Heesch, , Fig. 82.9]

Labelling Scheme: Topology & Combinatorics $\Rightarrow$ Combinatorics

Surface $\rightarrow$ Triangulation $\rightarrow$ Convex Polygon

Labelling Scheme: Topology & Combinatorics ⇒ Combinatorics

Surface → Triangulation → Convex Polygon



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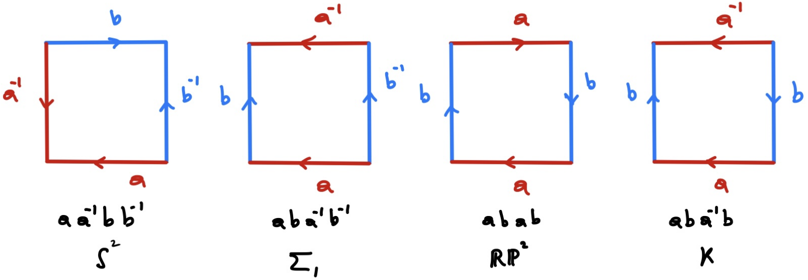
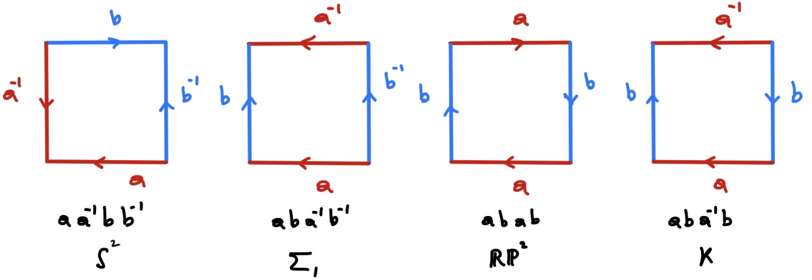


Question: How wild can such a labelling scheme look?

Labelling Scheme: Topology & Combinatorics ⇒ Combinatorics

Surface → Triangulation → Convex Polygon



Question: How wild can such a labelling scheme look?

Answer: Not very.

- Each label appears exactly twice
- with either positive or negative exponent.

Elementary Operations - Combinatorial Homeomorphism

Generic labelling scheme: $abda^{-1}cb^{-1}cd$

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The prime manifolds from the classification theorem have labelling scheme

$$\Sigma \sim aba^{-1}b^{-1}, \quad \mathbb{RP}^2 \sim abab$$

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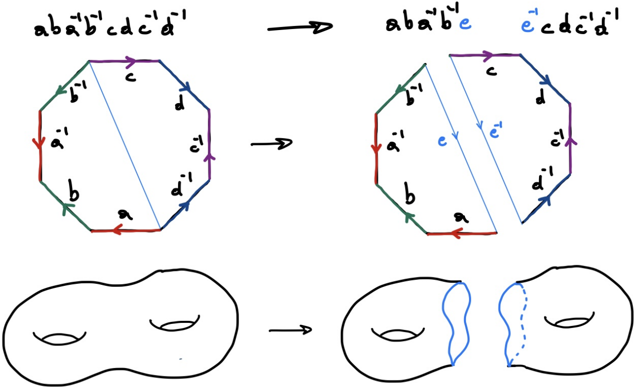
$$\Sigma \sim aba^{-1}b^{-1}, \quad \mathbb{RP}^2 \sim abab$$

Idea: Transform arbitrary labelling scheme into connected sum of primes (normal form), but keep homeomorphism type!

Elementary Operations - Combinatorial Homeomorphism

Elem. Operation	Example	Homeomorphism
cut & paste	$aba^{-1}b^{-1} \sim abc^{-1} \quad ca^{-1}b^{-1}$	connected sum
cyclic permutation	$aba^{-1}b^{-1} \sim b^{-1}aba^{-1}$	rotate polygon
formal inversion	$aba^{-1}b \sim b^{-1}ab^{-1}a^{-1}$	reflect polygon
re-labelling	$ababcc \sim ddcc$	-



Elementary Operations in Action

Example

Elementary operations can be used to show that $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$.

Proof.

$aba^{-1}b$ is a labelling scheme of K.

$$
\begin{aligned}
& aba^{-1}b, && \\
& bac \quad c^{-1}ba^{-1}, && \text{cycl. permutation \& cutting along } c \\
& acb \quad ab^{-1}c, && \text{cycl. permutation \& formal inversion} \\
& acb \quad b^{-1}ca, && \text{cycl. permutation} \\
& aacc, && \text{pasting along } b \text{ \& cycl. permutation} \\
& ababcdcd, && \text{re-labelling}
\end{aligned}
$$

which is a labelling scheme of $\mathbb{RP}^2 \# \mathbb{RP}^2$. □

Classification is minimal

Thus, we have a classification - but is it minimal?
Or could it be that

$$\Sigma_{42} \cong P_{197}$$

This cannot happen! We show that different members have different fundamental group.

Recall: Seifert–Van Kampen's Theorem

S a top. space. $S = A \cup B$, $A, B \subseteq S$ open, path-connected, $A \cap B$ path-connected. Then

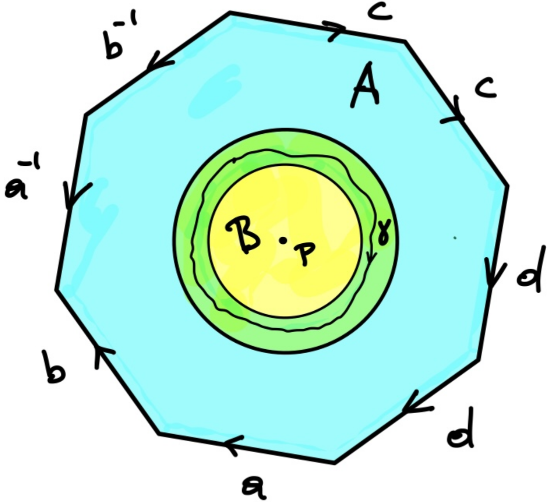
$$\begin{array}{ccc} \pi_1(A \cap B, p) & \xrightarrow{i_*} & \pi_1(A, p) \\ {\scriptstyle j_*}\downarrow & & \downarrow \\ \pi_1(B, p) & \longrightarrow & \pi_1(S, p) \end{array}$$

is a push-out of groups and

$$\pi_1(S, p) = \frac{\pi_1(A, p) * \pi_1(B, p)}{\langle i_*([\gamma]) j_*([\gamma])^{-1}, [\gamma] \in \pi_1(A \cap B, p) \rangle}$$

Members of the Family are distinct

Consider the labelling scheme $aba^{-1}b^{-1}ccdd$.



Members of the Family are distinct

Seifert–Van Kampen gives the following pushout diagram

$$\begin{array}{ccccc} \pi_1(A \cap B) & \xrightarrow{i_*} & \pi_1(A) & \qquad & [\gamma] \mapsto [aba^{-1}b^{-1}ccdd] \\ {\scriptstyle j_*}\downarrow & & \downarrow & & \downarrow \\ \pi_1(B) & \longrightarrow & \pi_1(S) & & e \end{array}$$

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\end{array}
\qquad\qquad
\begin{array}{c}
[\gamma] \mapsto [aba^{-1}b^{-1}ccdd] \\
\big\downarrow \\
e
\end{array}
$$

$$\Rightarrow \pi_1(S) = \langle a, b, c, d | aba^{-1}b^{-1}ccdd \rangle$$

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$$\Rightarrow \pi_1(S) = \langle a, b, c, d | aba^{-1}b^{-1}ccdd \rangle$$

For the prime manifolds:

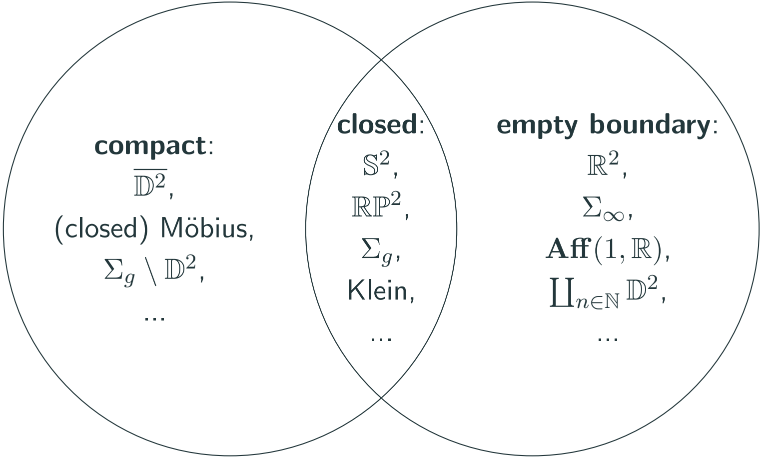
$$\mathsf{Ab}(\pi_1(\Sigma_g)) \cong \mathbb{Z}^{2g} \quad \mathsf{Ab}(\pi_1(P_m)) \cong \mathbb{Z}^{m-1} \times \mathbb{Z}/2\mathbb{Z}$$

So differ by number of generators and torsion.

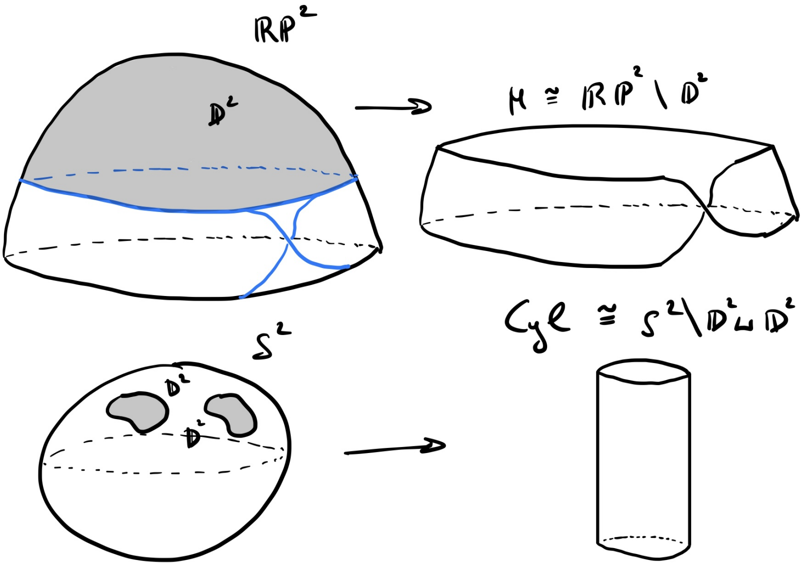
Review

Generalizations and Dimension ≥ 3

Recall: Further Subdivision



Surfaces with Boundary



Surfaces with Boundary

Theorem ([Munkres, 2000], §78, Exercise 5)
Any surface with boundary is homeomorphic to a surface without boundary with a disc removed.

Since $S \setminus \mathbb{D}^2 \cong S\#\mathbb{D}^2$ there is a bijection

$$\{\text{cpt. surfaces, } \partial S \neq \emptyset\}/\{\text{homeo}\} \to \langle \Sigma_1, \mathbb{RP}^2, \mathbb{D}^2 | P_3 \cong \mathbb{RP}^2\#\Sigma \rangle$$

Non-Compact Surfaces

More complicated:

- $[0,1]^2 \setminus C$, where C is the Cantor set
- Σ_∞ surface of infinite genus
- ...

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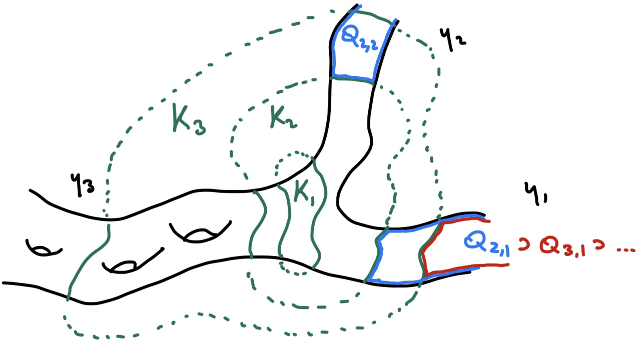
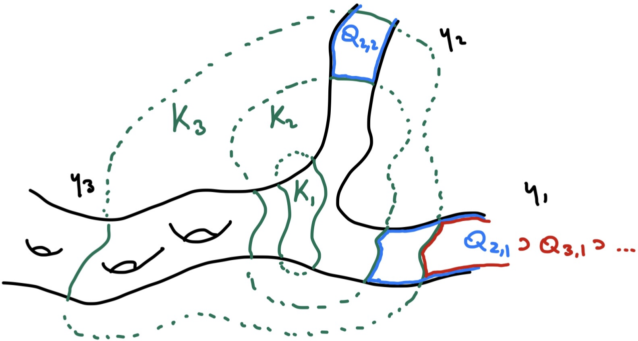
- ...

Idea: Exhaust S by compact sets $K_1 \subseteq K_2 \subseteq \ldots$ and study each compact set separately.

To control behaviour at infinity, study sequences $Q_{1,i_1} \supseteq Q_{2,i_2} \supseteq Q_{3,i_3} \supseteq \ldots$ "ends" of connected components of the complements of the K_n that do not terminate.

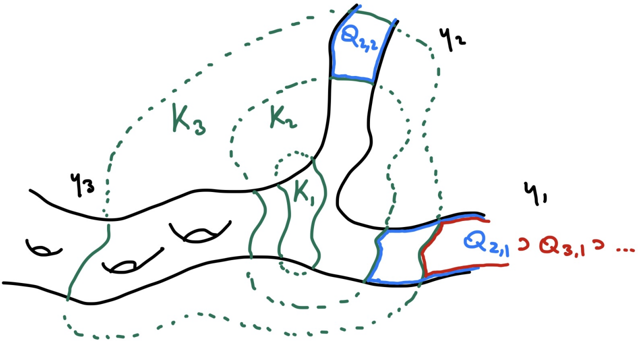
Non-Compact Surfaces





$\Rightarrow$ 4 types of orientability, possibly infinite genus as invariants.

Non-Compact Surfaces



⇒ 4 types of orientability, possibly infinite genus as invariants.

Theorem: If these invariants coincide and the ends are equivalent, then two surfaces are homeomorphic.

This generalizes the classification theorem for compact surfaces, which do not have ends.

Dimension ≥ 3

Considerably more complicated, but ...

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Considerably more complicated, but ...

Theorem (Milnor)

Any connected, compact, orientable 3-manifold M is the unique connected sum of finitely many prime manifolds, where a manifold is prime if $M = M \# \mathbb{S}^3$ is its only such representation.

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