Classification of Surfaces

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1 Introduction

The classification problem for manifolds is a central topic in geometric topology and has been for the last 170 years. Its difficulty varies drastically with the dimension of the manifold: In dimension 0 it is trivial, in dimension 1 a reasonable exercise, in dimension 2 the topic of a TopMath admission essay, in dimension 3 it was (largely) resolved around the turn of the millennium, and in dimension ≥ 4 it is an open problem. Before making any classification attempt we need to ask what a surface is and when two surfaces are equal. In more modern terms, we need to choose the right category to work in. The relevant candidates are

- **TopMan**, the category of topological manifolds and continuous maps,
- **PLMan**, the category of piecewise linear manifolds and piecewise linear maps, and
- C^∞**Man**, the category of smooth manifolds and smooth maps.

While this distinction is vital in higher dimension, in dimension ≤ 3 all three of these categories are equivalent, so we will gloss over these subtleties whenever possible.

2 Surfaces

It is our goal to find the isomorphism types of 2-manifolds i.e. of non-empty, second countable, locally Euclidean Hausdorff topological spaces (with or without boundary) of dimension 2. However, the following lemma lets us immediately reduce this to the classification of *connected* 2-manifolds.

Lemma 1 ([Lee13, Prop. 1.11])**.** *Let M be a manifold (with or without boundary) of dimension n. Then the set of (path-)connected*[1] *components $\pi_0(M)$ is countable, and every (path-)connected component is open, closed, and itself an n-manifold (with or without boundary). Furthermore $M \cong \coprod_{N \in \pi_0(M)} N$ and if M is compact $\pi_0(M)$ is finite.*

With this in mind, for the purpose of this essay, we will assume that a surface is connected and consider the further subdivision:

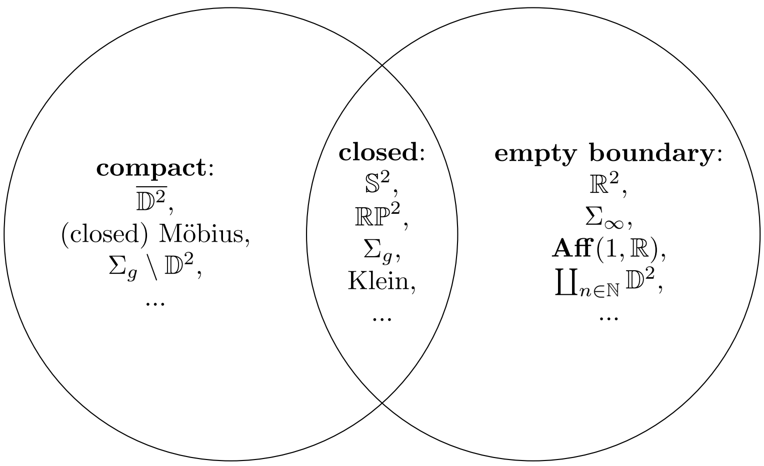

Figure 1: Further subdivision of surfaces.

Since compactness and empty boundary are topological properties, those three classes are closed under homeomorphism, and thus the classification can be further organized into focusing on each of the classes individually. The most well-behaved of these is that of closed (i.e. compact & without boundary) surfaces, which includes the 2-sphere, the torus, and projective 2-space. For this class, we will sketch a proof of classification below.

In section 4 we will make some remarks about the remaining classes, which are also fully classified. A detailed account can be found in [Lal14, Ch. 5, 6].

[1] Due to being locally Euclidean and thus locally path-connected the two notions of connectedness and path-connectedness coincide for manifolds.

3 Classification of Closed Surfaces

While there are many ways to prove the classification, we choose a combinatorial approach and work in **TopMan**. Working in the category C^∞**Man** yields more refined tools such as gradient-like flow via Morse theory, an account of which is given in [Hir94, Ch. 9] and [Lor10]. However, as opposed to the explicit combinatorial construction in our proof, the Morse-theoretic approach generalizes well to higher dimensions, ultimately manifesting itself in surgery-theoretic methods.

Theorem 1. *(Classification of Closed Surfaces) Let S be a closed surface. Then S is homeomorphic to exactly one member of the following families:*

- *the* 2***-sphere*** $\mathbb{S}^2$ *(=: Σ_0)*
- *the* ***genus*** g***-surface*** $\Sigma_g := \underbrace{\mathbb{T}^2\#\ldots\#\mathbb{T}^2}_{g\ \text{times}}$ *("surface of torus type")*
- *the* m***-fold projective plane*** $P_m := \underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{m\ \text{times}}$ *("surface of projective type")*

Furthermore,

$$P_3 \cong \mathbb{RP}^2\#\Sigma_1 \tag{1}$$

is the only non-trivial relation.

Here # denotes the connected sum of two surfaces[2] M, N, which is the glueing of M and N along the boundary of embeddings of discs $i, j : \mathbb{D}^2 \hookrightarrow M, N$ i.e.

$$M\#N := M \setminus i(\mathbb{D}^2) \coprod_{\partial i(\mathbb{D}^2)\cong \partial j(\mathbb{D}^2)} N \setminus j(\mathbb{D}^2)\ . \tag{2}$$

Before proving the theorem, we observe the special case in which a closed surface is homeomorphic to the quotient of a convex polygon in $\mathbb{R}^2$. For example, the torus can be given as the quotient of a square as indicated in figure 3.

Formally, we observe that there is a homeomorphism between Σ_1 and the quotient space of $[0,1]^2$ via the quotient map π satisfying

$$\forall t \in [0,1] : \pi(0,t) = \pi(1,t) \quad \text{and} \quad \pi(t,0) = \pi(t,1)\ , \tag{3}$$

i.e. by identifying edges. The first condition identifies the edges a and a^{-1}, while the second identifies b with b^{-1}.

In fact, this is not a special case at all, as we will see that any closed surface is homeomorphic to a quotient of a polygon (we call this a **surface representation**). This is the first step in the proof. In the second step we will show that each surface representation gives a member of one of the families above. Lastly, we want to check that S is homeomorphic to *exactly* one member of the families i.e. that no member in those families is homeomorphic to any other - this will be done by considering their fundamental group.

Triangulation: Topology ⇒ Topology & Combinatorics. Showing that any surface has a surface representation is highly technical and beyond the scope of this essay. However, we will go over a rough sketch of the ideas involved and provide references to detailed accounts.

Given a surface S, one can find a space which is obtained by glueing together triangles (simplicies) at their boundaries, to which it is homeomorphic - this is called a **triangulation** of S (see figure 4). It is a more combinatorial model, which lends itself much better to classification. In particular, a triangulation of a surface can be "cut open" and "flattened out" to obtain a surface representation (e.g. cut open figure 4 along the paths a and b in figure 3). However, whether a manifold has a triangulation or not is a highly non-trivial problem. In dimension ≤ 3 this is indeed the case as was shown in [Rad25][3] (for dim 2) and

[2]For a 2-dimensional example see figure 7. Note that a priori it is not clear at all (and without further consideration of orientability not even true in higher dimension) that this gives a well defined manifold (even only up to homeomorphism) i.e. that $N\#M$ is independent of the choice of i, j as well as the identification at the boundary. In the topological category **TopMan** the proof relies on the Annulus Theorem, which very roughly says that the difference between two nested topological n-spheres is an annulus. In dimension 2 this was already proven in [Rad25], but in general dimension, in particular for dim ≥ 5, the proof was only completed in 1969 in [Kir10]. Further discussion as well as comments on the smooth setting C^∞**Man** can be found in [(ht].

[3]A more modern account is given in [GX12, Appx. E].

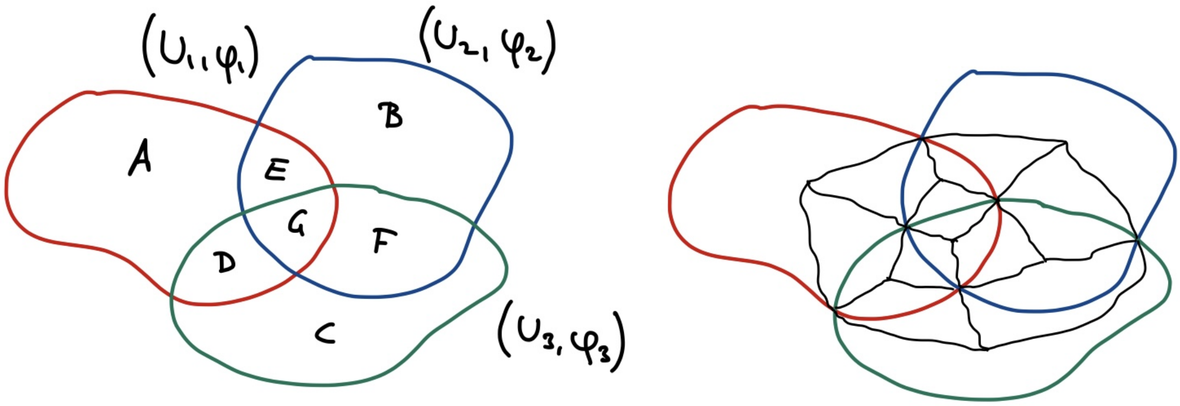

Figure 2: Subdivision of Jordan regions by intersecting with overlapping regions.

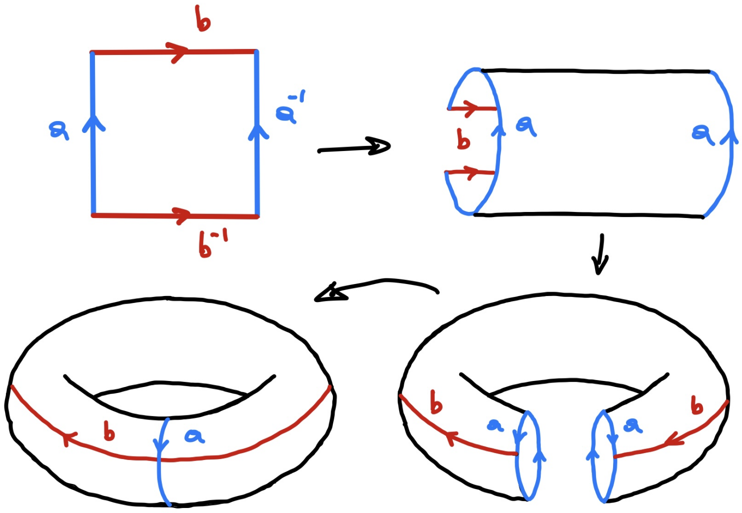

Figure 3: Torus as a quotient of a convex polygon.

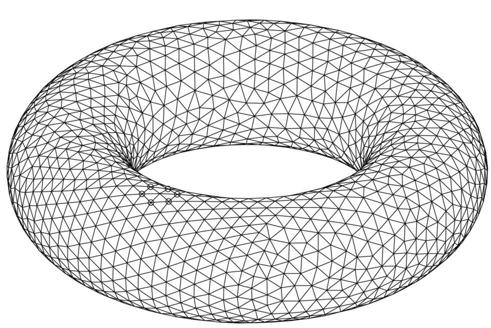

Figure 4: A triangulation of the torus.[5]

[Moi52, Ch. 23] (for dim 3). However, in higher dimensions this is not true: for dim $= 4$ a counterexample is given in [AM90], and a proof for the existence of counterexamples in dimension ≥ 5 in [Man16].

For $\mathbb{D}^2$ there clearly exist triangulations. So, since surfaces are locally Euclidean, the construction comes down to successively choosing triangulations of the Euclidean charts that are compatible in the intersection with other charts, so as to form a global triangulation.

For this, pick some covering of S by coordinate charts $\{(\varphi_i, U_i)\}_{i \in I}$ such that each φ_i can be extended to $\overline{U_i} \subseteq S$, each U_i intersects only finitely many others, and the intersections $\partial U_i \cap \partial U_j$ of the boundaries of any two chart-domains consists of at most finitely many points or arcs. This is called a **Jordan covering of finite character**, and any surface admits one - cf. [Lal14, Thm. 2.3.6.].

Now, for any $j \neq 1$, the intersection $\partial U_1 \cap U_j$ divides U_j into Jordan regions, which is a consequence of the Jordan–Schoenflies theorem. This is another technical theorem, a precise formulation and proof of which is given in [Moi13, Ch. 9]. For example, in figure 2, the boundary $\partial U_1 \cap U_2$ divides U_2 into the two regions $B \cup F$ and $E \cup G$. Doing this for all charts yields a partition of S into Jordan regions, each of which has finitely many points on its boundary at which three or more regions meet. As each of the (closed) Jordan regions is homeomorphic to the (closed) unit disc, each can be triangulated by connecting one point to the points of intersection at the boundary[4] (cf. figure 2). Hence there exists a triangulation of the entire surface. A much more detailed description can be found in [Lal14, Sec. 2.3].

[4] Here we need the first assumption about Jordan regions to ensure that the triangulations agree at the boundary. The second and third assumption ensure that there are only finitely many points of intersection.

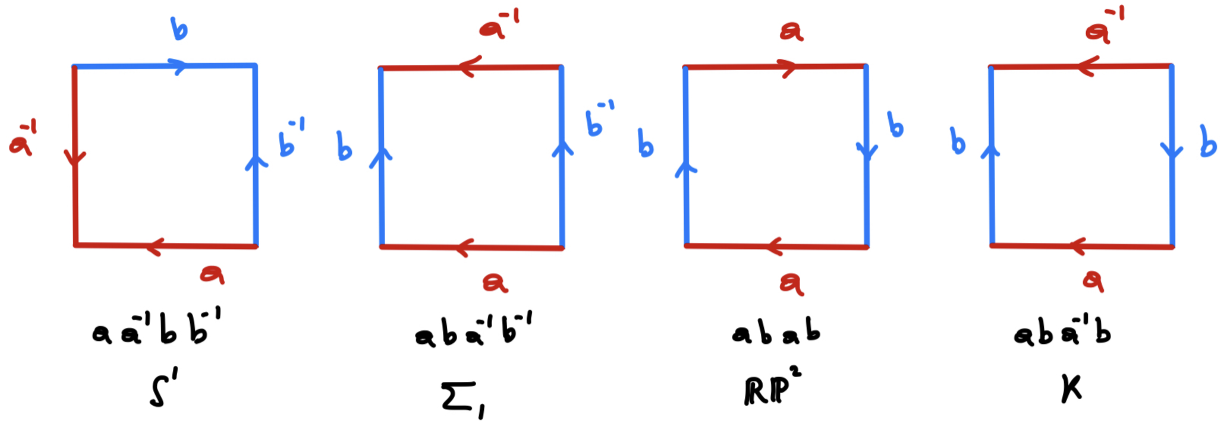

Figure 5: Labelling schemes and associated polygons for the sphere, torus, real projective plane, and Klein bottle.

Labelling scheme: Topology & Combinatorics $\Rightarrow$ Combinatorics. Just as for the torus, a general surface representation is completely specified by a labelling of its edges indicating how they are glued together. More formally, to a surface we can associate a formal word in which any letter is signed and appears exactly twice (which we will call a **labelling scheme**). The letters correspond to the edges of the polygon, the position in the word to their position in the polygon, and the sign to their orientation (relative to clockwise orientation about the center of the polygon). The fact that every edge appears exactly twice stems from the fact that any edge is obtained by cutting. Hence the two appearances correspond to the two boundaries of the cut.

Examples of surfaces, their surface representation and their labelling schemes are given in figure 5.

Elementary Operations. The combinatorial analogue of homeomorphisms of surfaces are the elementary operations. With their help one is able to determine the homeomorphism type of a surface representation and thus of a surface. Examples of those are given in table 1. Figure 7 illustrates the close relation between the two.

Elem. Operation	Example	Homeomorphism
cut & paste	$aba^{-1}b^{-1} \sim abc^{-1} \quad ca^{-1}b^{-1}$	identifying as conn. sum
cyclic permutation	$aba^{-1}b^{-1} \sim b^{-1}aba^{-1}$	rotate polygon
formal inversion	$aba^{-1}b \sim b^{-1}ab^{-1}a^{-1}$	reflect polygon
re-labelling	$ababcc \sim ddcc$	-

Table 1: Elementary operations on labelling schemes.

Example 1. Elementary operations can be used to show that $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$.

Proof. A surface representation and labelling scheme of the Klein bottle K given in figure 5. We now apply the following elementary operations to obtain a homeomorphic surface.

$$\begin{aligned}
&aba^{-1}b, && \\
&bac \quad c^{-1}ba^{-1}, && \text{cycl. permutation \& cutting along } c \\
&acb \quad ab^{-1}c, && \text{cycl. permutation \& formal inversion} \\
&acb \quad b^{-1}ca, && \text{cycl. permutation} \\
&aacc, && \text{pasting along } b \text{ \& cycl. permutation} \\
&ababcdcd, && \text{re-labelling}
\end{aligned}$$

The last labelling scheme belongs to a connected sum of two copies of $\mathbb{RP}^2$. $\square$

The relation $P_3 \cong \mathbb{RP}^2 \# \Sigma_1$ from theorem 1 can be shown in a very similar manner and the proof uses the above example (cf. [Lee10, Lem. 6.17]).

[5] https://en.wikipedia.org/wiki/Triangulation_(topology)#/media/File:Torus-triang.png

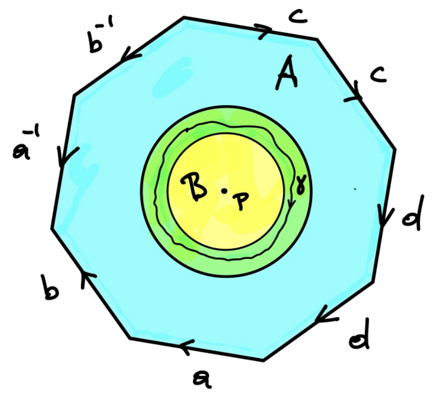

Figure 6: A surface representation of $aba^{-1}b^{-1}ccdd$.

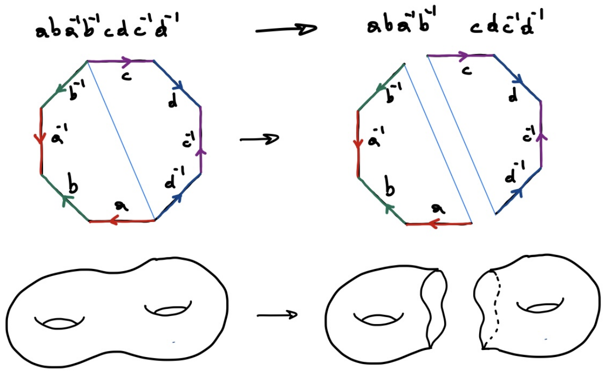

Figure 7: Cut & paste and connected sum.

Members of the families are distinct. In order to distinguish the different surfaces, we want to show that their fundamental groups differ[6]. We will illustrate this by means of an example.

Consider a surface representation of a surface S with labelling scheme $aba^{-1}b^{-1}ccdd$ as in figure 6, and a covering via an open disc (yellow, green) B and (a quotient of) an annulus A (blue, green) with some overlap $A \cap B$ (green). Then one can apply the Seifert–Van Kampen theorem to obtain the fundamental group of the resulting surface as the pushout of the fundamental groups of the two members of the covering and their intersection. To get an explicit description of $\pi_1(S)$, choose a generator $[\gamma]$ of

$$\pi_1(A \cap B) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z} \tag{4}$$

and chase it through the pushout diagram $(*)$. The class $[\gamma]$ necessarily has a representative γ winding around the point p once (w.l.o.g. clockwise). Now following the homomorphism i_* induced by the inclusion $A \cap B \hookrightarrow A$ and the deformation retract $\vee_{i=1}^4 \mathbb{S}_i^1 \hookrightarrow A$ we obtain the mapping $[\gamma] \mapsto [aba^{-1}b^{-1}ccdd]$ where each edge of the polygon is considered as a path in A. Hence, by definition of the pushout, $\pi_1(S) = \langle a, b, c, d \mid aba^{-1}b^{-1}ccdd\rangle$ i.e. the free group on 4 generators subject to the condition $e = aba^{-1}b^{-1}ccdd$, where e is the identity element of the group. Notice how in equation 4 we used the homotopy invariance of the fundamental group in the first isomorphism. The latter isomorphism can be proven by using covering spaces. This is done for example in [Lee10, Ch. 8].

$$\begin{tikzcd} A \cap B \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & A \arrow[d, hook] \\ B \arrow[r, hook] & S \end{tikzcd} \qquad \begin{tikzcd} \pi_1(A \cap B) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(A) \arrow[d] \\ \pi_1(B) \arrow[r] & \pi_1(S) \end{tikzcd} \qquad \begin{tikzcd} {[\gamma]} \arrow[r, mapsto] \arrow[d, mapsto] & {[aba^{-1}b^{-1}ccdd]} \\ e & \end{tikzcd} \tag{$*$}$$

In particular, this gives $\pi_1(\Sigma_1) \cong (\mathbb{Z} * \mathbb{Z})/(aba^{-1}b^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/(c^2) \cong \mathbb{Z}/2\mathbb{Z}$. More generally, the abelianization of the fundamental group satisfies[7]

$$\begin{aligned} \mathrm{Ab}(\pi_1(\mathbb{S}^2)) &\cong 0 \\ \mathrm{Ab}(\pi_1(\Sigma_g)) &\cong \mathbb{Z}^{2g} \\ \mathrm{Ab}(\pi_1(P_m)) &\cong \mathbb{Z}^{m-1} \times \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

Thus, the Σ_g, $g \geq 1$, and the P_m, $m \geq 1$, are among themselves distinguished by the number of generators of $\mathrm{Ab}(\pi_1)$, while for any $g, m \in \mathbb{N} : \Sigma_g \not\cong P_m$, as only $\mathrm{Ab}(\pi_1(P_m))$ contains a torsion element.

In fact, since we only consider the abelianization of π_1, by the Hurewicz isomorphism $\mathrm{Ab}(\pi_1(S)) \cong H_1(S)$, singular homology would be enough to distinguish the spaces. A proof via the latter would be very similar, using homotopy invariance and the Mayer–Vietoris long exact sequence as a substitute to the Seifert–Van Kampen theorem.

[6] Since we assumed surfaces to be path connected we omit a base point and consider the resulting groups only up to isomorphism.

[7] cf. [Lee10, Prop. 10.21]

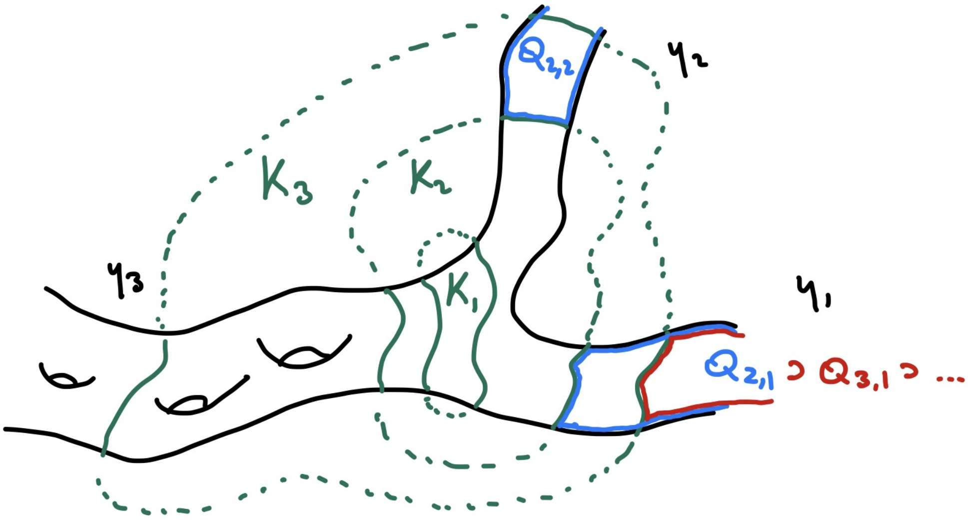

Figure 8: Compact exhaustion & ends

4 Generalizations

4.1 Compact Surfaces with Boundary

Using algebraic terms, the classification theorem of closed surfaces (with some additional steps) implies that the set of homeomorphism types, together with the connected sum operation and $\mathbb{S}^2$ as the identity element, forms a commutative monoid with generators $\Sigma_1, \mathbb{RP}^2$, subject only to the condition $P_3 \cong \mathbb{RP}^2 \# \Sigma_1$.

The classification of compact surfaces with boundary can be stated in much the same way. This is due to the fact that every compact surface with boundary can be given as a surface without boundary with open discs removed. For example the closed unit disc $\overline{\mathbb{D}^2} \cong \mathbb{S}^2 \setminus \mathbb{D}^2$ and the Möbius strip $M \cong \mathbb{RP}^2 \setminus \mathbb{D}^2$ (cf. [Mun00, §78, Exercise 5]). This leads to a similar monoid as before, but with a 2-disc as an additional generator satisfying no further relations. This was suggested in [hbb] by Tom Church.

4.2 Non-Compact Surfaces

Non-compact surfaces are more intricate in nature due to their "behaviour at infinity" (like a surface of infinite genus) or their "infinitesimal behaviour" (like the complement of the 1-dimensional Cantor set in $\mathbb{R}^2$). At the end of the day both phenomena are due to the same defect of the surface: non-compactness. To control such phenomena, one may choose a compact exhaustion $\cup_{n=1}^{\infty} K_n$ of a surface S. Define for each $n \in \mathbb{N}$ the connected components of $(K_n)^{\complement}$ as $Q_{n,j}$. Then we consider descending sequences $\eta_i := Q_{1,i_1} \supseteq Q_{2,i_2} \supseteq Q_{3,i_3} \supseteq \dots$ (called **ends**) to detect unbounded parts of the surface. Ultimately, this leads to 4 types of orientability and a possibly infinite genus g as invariants of a non-compact surface. Then, if the genus and orientability type of two surfaces coincide, they are homeomorphic if and only if the aforementioned ends coincide (in a suitable sense). Notice that this nicely incorporates the classification for compact surfaces, which are completely classified by their orientability and genus because they have no ends.

The case of non-compact surfaces with boundary does not look very different than the analogous situation in the compact case. A more detailed explanation of the above can be found in [Lal14, Ch. 6].

4.3 Dimension > 2

As already suggested, the problem is considerably more complicated in higher dimension.

The fundamental structure theorem for dimension 3, due to Milnor [Mil62], says that any (connected, compact, orientable) 3-manifold is the unique connected sum of finitely many prime manifolds, where we call a manifold M **prime** if $M = M \# \mathbb{S}^3$ is its only such representation i.e. if it cannot be decomposed any further. After managing some subtleties regarding the well-definedness of the connected sum operation (cf. footnote 2), this reduces the classification to that of the prime manifolds. Allen Hatcher gives a short account in [Hat]. In [hba] there is a discussion about a classifying monoid of higher dimensional manifolds, as introduced for surfaces in subsection 4.1.

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