Fractional Brownian Motion

Gideon Chiusole

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Abstract

Fractional Brownian motion (fBM) is a one parameter generalization of Brownian motion which can be seen as the convolution of white noise with a power kernel $t^{H-\frac{1}{2}}$, splitting fBM into three quite distinct classes: $0 < H < \frac{1}{2}$, $H = \frac{1}{2}$, and $\frac{1}{2} < H < 1$. Originally, fBM was introduced by B. Mandelbrot and J. Van Ness as a continuous time model for a long-range dependent stochastic process, specifically for the study of economics, hydraulics, and fluctuation in solids. From a probabilistic point of view, fBM is particularly interesting since it is neither a Markov process nor a semi-martingale. We will show both of these results alongside some other probabilistic and analytic properties.

Contents

1 Motivation - Rough Volatility

In a first model for the behaviour of derivatives prices, the value of an underlying asset S is assumed to satisfy

$$\mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}B_t, \quad t \geq 0,$$

where $(S_t)_{t\geq 0}$ is the price of the asset, $(B_t)_{t\geq 0}$ a Brownian motion, $(\mu_t)_{t\geq 0}$ is a drift, and $(\sigma_t)_{t\geq 0}$ is the volatility of the asset. Classically, σ is assumed to be constant or at least deterministic (e.g. Black-Scholes or Cox–Ross–Rubinstein). A better model is provided by assuming σ itself to be random, leading to so-called stochastic volatility model (e.g. Hull-White, Heston, SABR, CEV, GARCH, ...). Now the volatility itself satisfies

$$\mathrm{d}\sigma_t = f(t, \sigma_t)\mathrm{d}t + g(t, \sigma_t)\mathrm{d}B'_t, \quad t \geq 0,$$

where $f, g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable and $(B'_t)_{t\geq 0}$ is another Brownian motion (possibly related to $(B_t)_{t\geq 0}$). While such approaches give better models, there is (recent) evidence that Brownian motion is not the best choice for the driver of volatility (see [7, Sec. 2]). Across many markets and asset classes one can observe that for a time lag $\Delta \geq 0$

$$\mathbb{E}[|\ln(\sigma_{t+\Delta}) - \ln(\sigma_t)|^\alpha] \approx C\Delta^{\alpha H},$$

where H is not equal to 0.5 (as for Brownian motion), but rather $H \approx 0.13$. If we now suppose that the driver of log-volatility is continuous and has stationary increments (which are both very reasonable assumptions), we immediately arrive at fractional Brownian motion (fBM) with Hurst parameter $H \approx 0.13$ as a suitable driver.

2 Recap on Stochastic Processes

Recall that a **stochastic process** is a collection of random variables $X = (X_t)_{t\in T}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and indexed by an indexing set T, which we will call **time**. In the following, T will usually be $\mathbb{N}$, $\mathbb{R}$, $[0, \infty)$ or $[0, 1]$; although, in the introduction, any polish space (separable and complete metric space) is sufficient. A stochastic process is called **Gaussian** if for every $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ has a Gaussian distribution on $\mathbb{R}^n$. The collection of distributions of these random vectors is called the set of **finite dimensional distributions of** X. The functions

$$\begin{aligned} T &\to \mathbb{R} \\ t &\mapsto \mathbb{E}\big[X_t\big] \end{aligned} \qquad \text{and} \qquad \begin{aligned} T \times T &\to \mathbb{R} \\ (s,t) &\mapsto \operatorname{Cov}\big(X_s, X_t\big) \end{aligned}$$

are called the **expectation** and **covariance structure**, respectively, and they characterize a Gaussian process, i.e. if two Gaussian processes have the same expectation and covariance structure, then they have the same finite dimensional distributions. This is because for any $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ has the characteristic function

$$\underbrace{\int_\Omega \exp\left\{i\sum_{j=1}^n X_{t_j}\xi_j\right\}\mathrm{d}\mathbb{P}}_{=\text{ char. function}} = \exp\left\{i\sum_{j=1}^n \xi_j\mathbb{E}[X_{t_j}] - \frac{1}{2}\sum_{1\le i,j\le n}\operatorname{Cov}\left(X_{t_i},X_{t_j}\right)\xi_i\xi_j\right\}, \quad \xi\in\mathbb{R}^n$$

and the characteristic function of random vector characterizes its distribution uniquely. We will use the following notation interchangeably

$$\big[X_t\big](\omega) \equiv X(\omega,t) \equiv X_t(\omega), \quad \omega\in\Omega, t\in T.$$

If either of the two arguments of X is missing, we mean the function taking that argument as its input, i.e.

$$\begin{aligned} X(\omega): T &\to \mathbb{R} \\ t &\mapsto X_t(\omega) \end{aligned} \qquad \text{and} \qquad \begin{aligned} X_t: \Omega &\to \mathbb{R} \\ \omega &\mapsto X_t(\omega) \end{aligned}$$

2.1 Notions of Equivalence

As opposed to ordinary random variables, for stochastic processes, there are multiple reasonable notions of equivalence. Two stochastic processes $X = (X_t)_{t\in T}$, $Y = (Y_t)_{t\in T}$, defined on the same probability space $(\Omega,\mathbb{P})$ are said to

(1) **have the same finite dimensional distributions**, if for every $n\in\mathbb{N}$ and $t_1,\dots,t_n\in T$ the random vectors $(X_{t_1},\dots,X_{t_n})$ and $(Y_{t_1},\dots,Y_{t_n})$ have the same distribution; we write $X\sim Y$.[1] This notion also applies if the two processes are defined on different probability spaces.

(2) **be modifications of each other**, if for every $t\in T: \mathbb{P}\left(\omega\in\Omega: X_t(\omega)=Y_t(\omega)\right)=1$.

(3) **be indistinguishable**, if $\mathbb{P}\big\{\omega\in\Omega: X(\omega)=Y(\omega)\big\}=1$.

In general, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). A counter-example showing (1) $\not\Rightarrow$ (2) is given by two independent Brownian motions. A counter-example to (2) $\not\Rightarrow$ (3) is the following.

Example 2.1 ((2) $\not\Rightarrow$ (3))**.** Let A be a random variable with exp(1)-distribution, let $0=(0)_{t\ge0}$ be the constant 0-process, and define $Y=(Y_t)_{t\ge0}$ by

$$Y_t(\omega) = \begin{cases} 1, & A(\omega)=t \\ 0, & A(\omega)\ne t. \end{cases}$$

Then for every $t\ge0$

$$\mathbb{P}\big\{Y_t=0\big\} = \mathbb{P}\big\{\omega\in\Omega: A(\omega)\ne t\big\} = 1,$$

[1] We will use the same symbol to denote that two functions are asymptotically equivalent whenever the difference is clear.

where in the last equality we used the fact that A is exp(1)-distributed and thus has density w.r.t. the Lebesgue measure on $\mathbb{R}$. Hence Y is a modification of X. However, since for every ω there exists a unique $t \geq 0$ s.t. $A(\omega) = t$, every sample path of Y is non-zero at some $t \geq 0$. Thus

$$\mathbb{P}(Y_t = 0, \forall t \geq 0) = 0.$$

In this example, X had continuous sample paths $\mathbb{P}$-a.s. and Y had discontinuous sample paths $\mathbb{P}$-a.s., showing that two processes may have drastically different sample paths, despite being modifications of each other. However, as the next proposition shows, if the two processes are continuous, this cannot happen.

Proposition 2.2. *Let $X = (X_t)_{t\in T}, Y = (Y_t)_{t\in T}$ be two stochastic processes with $\mathbb{P}$-a.s. continuous sample paths that are modifications of each other i.e. for every $t \in T : \mathbb{P}(X_t = Y_t) = 1$. Then X and Y are indistinguishable.*

Proof. Let S denote a dense countable subset of T, which exists since we assumed S to be Polish. Then since X and Y are continuous

$$\{\omega \in \Omega : X(\omega) = Y(\omega)\} = \bigcap_{t\in T} \{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\} = \bigcap_{t\in S} \{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\}.$$

Since X and Y are modifications of each other, each set on the right hand side has full measure, and since S is countable, the intersection has full measure too. □

Corollary 2.3. *If a process X has a continuous modification $\widetilde{X}$, then this modification is unique among modifications (up to indistinguishably).*

On $\mathbb{R}$, both Proposition 2.2 and 2.3 can be generalized to cadlag functions, since two cadlag functions that agree on $\mathbb{Q}$ agree on $\mathbb{R}$.

2.2 Hermite Polynomials

Let $(X_t)_{t\geq 0}$ be a Gaussian process and A a real-valued $\mathcal{N}(0,1)$-distributed random variable. For the study of functionals of A, i.e. random variables of the form $F(A)$, where $F : \mathbb{R} \to \mathbb{R}$ is a measurable function, it is very convenient to study the space $L^2(R, \gamma_1)$ where $\gamma_1 = \mathcal{N}(0,1)$ is the standard normal distribution. This is because $F(A)$ lies in $L^2(\Omega, \mathbb{P})$ precisely if F lies in $L^2(\mathbb{R}, \gamma_1)$. To see this, note that

$$\|F(A)\|^2_{L^2(\Omega,\mathbb{P})} = \mathbb{E}_{\mathbb{P}}[F(A)^2] = \int_\Omega F(A(\omega))^2 \mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}} F(x)^2 \mathrm{d}\gamma_1(x) = \mathbb{E}_{\gamma_1}[F^2] = \|F\|^2_{L^2(\mathbb{R},\gamma_1)}.$$

Remark 2.4. But what about functionals of a $\mathcal{N}(0,\sigma^2)$-distributed random variable A when $\sigma^2 \neq 1$? In that case $F(A)$ is square integrable precisely when $F(\frac{1}{\sigma}\cdot)$ lies in $L^2(\mathbb{R}, \gamma_1)$. But what about functionals of a random vector $(X_{t_1}, \dots, X_{t_n})$? There are very similar constructions on $\mathbb{R}^n$.

Consider the operator $\tilde{\partial} : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$, defined by

$$\left[\tilde{\partial}(\phi)\right](x) = x\phi(x) - \phi'(x), \quad \phi \in C_c^\infty(\mathbb{R}), x \in \mathbb{R}.$$

It is the adjoint of the usual linear differentiation operator $\phi \mapsto \phi'$ under the inner product on $L^2(\mathbb{R}, \gamma_1)$ (not $L^2(\mathbb{R}, \lambda)$!). The set of Hermite polynomials $(H_k)_{k\in\mathbb{N}}$ is then defined by

$$H_0 \equiv 1 \quad \text{and} \quad H_k = \tilde{\partial}^k(H_0) \;\text{ for every }\; k \in \mathbb{N}.$$

Lemma 2.5. *Let $(H_k)_{k\in\mathbb{N}}$ denote the Hermite polynomials on $\mathbb{R}$. Then*

1. $\left(\frac{H_k}{\sqrt{k!}}\right)_{k\in\mathbb{N}}$ *is an orthonormal basis of $L^2(\mathbb{R}, \gamma_1)$.*
2. *for any Gaussian random vector (U, V) with $U \sim V \sim \mathcal{N}(0, 1)$ and any $k, l \in \mathbb{N}$ we have*

$$\mathbb{E}[H_k(U)H_l(V)] = \begin{cases} k!\mathbb{E}[UV]^k, & k = l \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. See [11, Prop. 1.3(2) & (3)]. □

3 Definition, Existence, and Basic Properties

3.1 Definition

In their original paper, B.B. Mandelbrot and J.W. Van Ness *define* fBM as (3.5) and then deduce properties (3.1) and (3.2). We take the opposite approach since it agrees better with our introduced motivation and was arguable the raison d'être for fBM (see [9, Sec. 3]).

Definition 3.1. A **fractional Brownian motion** with Hurst parameter $H \in (0, 1)$ is a continuous Gaussian process $B^H = (B^H)_{t\geq 0}$ which has stationary increments, i.e. for every $t_0 \geq 0$

$$\left(B^H_{t_0+t} - B^H_{t_0}\right)_{t\geq 0} \sim \left(B^H_t\right)_{t\geq 0}, \tag{3.1}$$

is H-self similar, i.e. for any $\alpha > 0$

$$\left(B^H_{\alpha t}\right)_{t\geq 0} \sim \left(\alpha^H B^H_t\right)_{t\geq 0}, \tag{3.2}$$

starts at 0 and is normalized, i.e.

$$B^H_0 = 0 \quad \mathbb{P} - a.s. \quad \text{and} \quad \operatorname{Var}\left[B^H_1\right] = 1.$$

This definition uniquely determines the expectation and covariance structure, and thus the finite dimensional distributions of B^H:

Let $t > 0$ be arbitrary. Then from stationarity, with $t_0 = t > 0$, we get

$$\mathbb{E}\left[B_{2t}^H - B_t^H\right] = \mathbb{E}\left[B_t^H\right] \quad \Rightarrow \quad \mathbb{E}\left[B_{2t}^H\right] = 2\mathbb{E}\left[B_t^H\right]$$

and from self-similarity

$$\mathbb{E}\left[B_{2t}^H\right] = 2^H\mathbb{E}\left[B_t^H\right] \quad \Rightarrow \quad 2^H\mathbb{E}\left[B_t^H\right] = 2\mathbb{E}\left[B_t^H\right].$$

Since $H \neq 1$, this implies $\mathbb{E}[B_t^H] = 0$ for every $t > 0$. Furthermore, $\mathbb{E}[B_0^H] = 0$ by assumption. For the covariance structure, let $s, t \geq 0$ be arbitrary. Then

$$\begin{aligned}
\mathbb{E}[B_t^H B_s^H] &= \frac{1}{2}\left(\mathbb{E}\left[(B_t^H)^2\right] + \mathbb{E}\left[(B_s^H)^2\right] - \mathbb{E}\left[(B_t^H - B_s^H)^2\right]\right) \\
&= \frac{1}{2}\left(\mathbb{E}\left[(t^H B_1^H)^2\right] + \mathbb{E}\left[(s^H B_1^H)^2\right] - \mathbb{E}\left[(B_{|t-s|}^H)^2\right]\right) \\
&= \frac{1}{2}\left(\mathbb{E}\left[(t^H B_1^H)^2\right] + \mathbb{E}\left[(s^H B_1^H)^2\right] - \mathbb{E}\left[(|t-s|^H B_1^H)^2\right]\right) \\
&= \frac{1}{2}\left(t^{2H}\underbrace{\mathbb{E}\left[\left(B_1^H\right)^2\right]}_{=1} + s^{2H}\underbrace{\mathbb{E}\left[\left(B_1^H\right)^2\right]}_{=1} - |t-s|^{2H}\underbrace{\mathbb{E}\left[(B_1^H)^2\right]}_{=1}\right) \\
&= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),
\end{aligned}$$

where for the first two terms we used self-similarity, and for the latter term we used stationarity of increments and then self-similarity. For the last line, we used the assumption of $\mathbb{E}\left[(B_1^H)^2\right] = 1$. In particular, we conclude that for $t \geq 0$

$$\mathbb{E}\left[B_t^H\right] = 0 \quad \text{and} \quad \operatorname{Var}\left[B_t^H\right] = t^{2H}. \tag{3.3}$$

In fact, we may extend this definition to include the case $H = 1$, where

$$B_t^1 = tB_1^1, \quad t \geq 0.$$

However, in the following we will still restrict ourselves to $0 < H < 1$.

We have seen that one can deduce the expectation and covariance structure from the defining features of fBM. Conversely, one can deduce the defining features (and some other basic properties) from the expectation and covariance structure.

Proposition 3.2. *Let $X = (X_t)_{t\geq 0}$ be a centred Gaussian process with*

$$\operatorname{Cov}(X_s, X_t) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

Then the following are true:

(1) For every $t \geq 0$ the random variable X_t has distribution $\mathcal{N}(0, t^{2H})$.

(2) Self similarity: for every $\alpha > 0 : (X_{\alpha t})_{t\geq 0} \sim (\alpha^H X_t)_{t\geq 0}$.

(3) Stationary increments: For every $t_0 \geq 0$, $(X_{t_0+t} - X_{t_0})_{t\geq 0} \sim (X_t)_{t\geq 0}$.

(4) Time inversion: $(t^{2H}X(1/t))_{t>0} \sim (X_t)_{t>0}$.

Proof. 1. For every $t \geq 0 : \mathbb{E}(X_t) = 0$ by definition. For the variance we have

$$\operatorname{Var}(B_t) = \operatorname{Cov}(X_t, X_t) = \frac{1}{2}\left(t^{2H} + t^{2H} - \underbrace{|t-t|^{2H}}_{=0}\right) = t^{2H}.$$

2. Let $\alpha > 0$. We want to show that the expectation and covariance structures of the two processes coincide. Let $s, t \geq 0$ be arbitrary. Then

$$\alpha^H \underbrace{\mathbb{E}\big[X_{\alpha t}\big]}_{=0} = 0 = \mathbb{E}\big[X_{\alpha t}\big]$$

and

$$\begin{aligned}
&\operatorname{Cov}\left(X_{\alpha s}, X_{\alpha t}\right) = \frac{1}{2}\left((\alpha t)^{2H} + (\alpha s)^{2H} - |\alpha s - \alpha t|^{2H}\right)\\
&= \alpha^{2H}\frac{1}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right) = \alpha^{2H}\operatorname{Cov}\left(X_s, X_t\right) = \operatorname{Cov}\left(\alpha^H X_s, \alpha^H X_t\right).
\end{aligned}$$

3. Let $t_0 > 0$ and $s, t \geq 0$ be arbitrary. Then

$$\mathbb{E}\big[X_{t_0+t} - X_{t_0}\big] = \mathbb{E}\big[X_{t_0+t}\big] - \mathbb{E}\big[X_{t_0}\big] = 0 + 0 = 0 = \mathbb{E}\big[X_t\big].$$

Substituting $t_1 = t_0 + t$, $t_2 = t_0 + s$, and $s_1 = s_2 = t_0$ into ((3.8)) (or making a short computation) gives

$$\begin{aligned}
&\operatorname{Cov}\left(B^H_{t_0+t} - B^H_{t_0}, B^H_{t_0+s} - B^H_{t_0}\right) =\\
&= \frac{1}{2}\Big[|t_0 - (t_0+s)|^{2H} + |(t_0+t) - t_0|^{2H} - |(t_0+t) - (t_0+s)|^{2H} - |t_0 - t_0|^{2H}\Big]\\
&= \frac{1}{2}\Big[|s|^{2H} + |t|^{2H} - |t-s|^{2H}\Big] = \operatorname{Cov}\left(B^H_t, B^H_s\right)
\end{aligned}$$

4. Omitted - similar to the above.

□

3.2 Existence

We have not yet shown that such a process exists in the first place. The usual way of going about this would be to establish the existence as a coordinate process on $\mathbb{R}^{[0,\infty)}$ and then using the Kolmogorov–Centsov Continuity Theorem (Thm. 4.1) to deduce that it has a version with continuous sample paths. However, we will choose a different route and explicitly give fBM as a Wiener functional w.r.t. two-sided Brownian motion.

We define **two-sided Brownian motion** $X = (X_t)_{t\in\mathbb{R}}$ via

$$\overline{B}_t := \begin{cases} B_t^{(1)}, & t \geq 0 \\ B_{-t}^{(2)}, & t < 0 \end{cases},$$

where $B^{(1)}$ and $B^{(2)}$ are two independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $\overline{B}$ is clearly not a martingale w.r.t. its generated filtration $(\mathcal{F}_t)_{t\in\mathbb{R}} := (\sigma\{X_s : s \leq t\})_{t\in\mathbb{R}}$ since, for instance,

$$\mathbb{E}[\overline{B}_1|\mathcal{F}_{-1}] = \mathbb{E}\left[B_1^{(1)}\middle|\sigma\left\{B_{-s}^{(2)} : s \leq -1\right\}\right] = \mathbb{E}[B_1] = 0 \neq \overline{B}_{-1}.$$

Thus an integral of the form $\int f(t)\mathrm{d}\overline{B}_t$ is not defined in the Ito sense. For functions $f \in L^2(\mathbb{R})$ we thus define

$$\int_{\mathbb{R}} f(t)\mathrm{d}\overline{B}_t := \int_0^\infty f(t)\mathrm{d}B_t^{(1)} + \int_0^\infty f(-t)\mathrm{d}B_t^{(2)}.$$

With this definition, $\int f(t)\mathrm{d}\overline{B}_t$ is well defined when $f \in L^2(\mathbb{R})$, the Ito isometry holds in the sense that for every $t \in \mathbb{R}$

$$\mathbb{E}\left[\left(\int_{-\infty}^t f(s)\mathrm{d}\overline{B}_s\right)^2\right] = \int_{-\infty}^t f(s)^2\mathrm{d}s,$$

and we have

$$\mathbb{E}\left[\int_{\mathbb{R}} f(s)\mathrm{d}\overline{B}_s\right] = 0. \tag{3.4}$$

However as opposed to the classical Ito integral, $(\int_a^\cdot f(s)\mathrm{d}\overline{B}_s)_{t\in\mathbb{R}}$ may not be a martingale - take for instance $f := 1_{[-1,1]}$.

Theorem 3.3. *(fBM as a Wiener functional) Let $(\overline{B}_t)_{t\in\mathbb{R}}$ be a two-sided Brownian motion and let $H \in (0,1)$. Then the stochastic process $X = (X_t)_{t\geq 0}$ defined by*

$$X_t := \frac{1}{c(H)} \int_{\mathbb{R}} (t-s)^{H-\frac{1}{2}} 1_{\{s<t\}} - (-s)^{H-\frac{1}{2}} 1_{\{s<0\}} \mathrm{d}B_s, \quad t \geq 0, \tag{3.5}$$

with

$$c(H)^2 := \frac{1}{2H} + \int_0^\infty \left[(1+w)^{H-\frac{1}{2}} - w^{H-\frac{1}{2}}\right]^2 \mathrm{d}w, \tag{3.6}$$

is continuous, starts at 0*, has expectation and covariance structure*

$$\mathbb{E}[X_t] = 0 \quad \textit{and} \quad \mathrm{Cov}(X_t, X_s) = \frac{1}{2}\left[t^{2H} + s^{2H} - (t-s)^{2H}\right], \quad 0 \leq s \leq t,$$

respectively, and is thus a fractional Brownian motion with Hurst parameter H.

Proof. Firstly, as a sum of two processes defined by Ito integrals, X is continuous, and setting $t = 0$ gives $X_0 = 0$ $\mathbb{P}$-a.s.

Secondly, we show that the integrals in (3.5) and (3.6) are well-defined i.e. that the integrand of (3.5) is square integrable and that (3.6) is finite. The integrand in (3.5) can be rewritten as

$$\underbrace{(t-u)^{H-\frac{1}{2}}1_{\{u<0\}} - (-u)^{H-\frac{1}{2}}1_{\{u<0\}}}_{=:f_t(u)} + \underbrace{(t-u)^{H-\frac{1}{2}}1_{\{0<u<t\}}}_{=:g_t(u)}.$$

Since f_t and g_t have disjoint support, this allows us to analyse them separately. The square integral of g_t is finite since $2(H-\frac{1}{2}) > -1$ for any choice $H \in (0,1)$. The square of f_t has the asymptotic[2]

$$\left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right]^2 \sim \left(H - \frac{1}{2}\right)^2 t^2(-u)^{2H-3}, \quad \text{as } u \to -\infty,$$

and is thus integrable since $2H - 3 < -1$. Setting $t = 1$ and applying the transformation $u \mapsto -u$, we deduce the finiteness of (3.6) from the square integrability of f_t.

Thirdly, we have $\mathbb{E}[X_t] = 0$ for every $t \geq 0$. Using the Ito isometry, we see that for any $0 \leq s < t$

$$\mathbb{E}[(X_t - X_s)^2] = \frac{1}{c(H)^2}\int_{\mathbb{R}} \left[(t-u)^{H-\frac{1}{2}}1_{\{u<t\}} - (s-u)^{H-\frac{1}{2}}1_{\{u<s\}}\right]^2 \mathrm{d}u.$$

Substituting $u = -(t-s)w + s$ (and $\mathrm{d}u = (t-s)\mathrm{d}w$) yields

$$\begin{aligned}
&= \frac{1}{c(H)^2}\int_{\mathbb{R}} \left[\big((t-s)(1+w)\big)^{H-\frac{1}{2}}1_{\{w>-1\}} - \big((t-s)w\big)^{H-\frac{1}{2}}1_{\{w>0\}}\right]^2 (t-s)\mathrm{d}w \\
&= \frac{(t-s)^{2H}}{c(H)^2}\int_{\mathbb{R}} \left[(1+w)^{H-\frac{1}{2}}1_{\{w>-1\}} - w^{H-\frac{1}{2}}1_{\{w>0\}}\right]^2 \mathrm{d}w \\
&= \frac{(t-s)^{2H}}{c(H)^2}\underbrace{\int_0^\infty \left[(1+w)^{H-\frac{1}{2}} - w^{H-\frac{1}{2}}\right]^2 \mathrm{d}w + \underbrace{\int_{-1}^0 (1+w)^{2H-1}\mathrm{d}w}_{=\frac{1}{2H}}}_{=c(H)^2} \\
&= (t-s)^{2H}.
\end{aligned}$$

The result now follows by

$$\mathbb{E}\left[X_t^2\right] = \mathbb{E}\big[(X_t - \underbrace{X_0}_{=0 \text{ a.s.}})^2\big] = |t|^{2H}, \quad t \geq 0, \tag{3.7}$$

and

$$\mathbb{E}\left[(X_t - X_s)^2\right] = \mathbb{E}\left[X_t^2\right] - 2\,\mathrm{Cov}(X_t, X_s) + \mathbb{E}\left[X_s^2\right].$$

[2]Proof omitted.

Lastly, $\mathrm{Var}(X_1) = 1$ follows from (3.7). □

Corollary 3.4. *Fractional Brownian motion exists.*

3.3 Increment Process

Alongside fBM itself, we define its **process of increments (with lag $\Delta \geq 0$)** by

$$\left(B_t^{H,\Delta}\right)_{t\geq 0} := \left(B_{t+\Delta}^H - B_t^H\right)_{t\geq 0}.$$

This process is still centred, but its covariance structure is more involved: Generally, let $t_1, s_1, t_2, s_2 \geq 0$. Then

$$\begin{aligned}
&\mathrm{Cov}\left(B_{t_1}^H - B_{s_1}^H, B_{t_2}^H - B_{s_2}^H\right) = \mathbb{E}\left[\left(B_{t_1}^H - B_{s_1}^H\right)\left(B_{t_2}^H - B_{s_2}^H\right)\right] \\
&= \mathbb{E}\left[B_{t_1}^H B_{t_2}^H\right] - \mathbb{E}\left[B_{t_1}^H B_{s_2}^H\right] - \mathbb{E}\left[B_{s_1}^H B_{t_2}^H\right] + \mathbb{E}\left[B_{s_1}^H B_{s_2}^H\right] \\
&= \frac{1}{2}\Big[|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H} - |s_1|^{2H} - |t_2|^{2H} + |s_1 - t_2|^{2H} \\
&\quad - |t_1|^{2H} - |s_2|^{2H} + |t_1 - s_2|^{2H} + |s_1|^{2H} + |s_2|^{2H} - |s_1 - s_2|^{2H}\Big] \\
&= \frac{1}{2}\Big[|s_1 - t_2|^{2H} + |t_1 - s_2|^{2H} - |t_1 - t_2|^{2H} - |s_1 - s_2|^{2H}\Big] \qquad (3.8)
\end{aligned}$$

For the increment process this gives

$$\begin{aligned}
&\mathrm{Cov}\left(B_s^{H,\Delta}, B_t^{H,\Delta}\right) = \mathrm{Cov}\left(B_{s+\Delta}^H - B_s^H, B_{t+\Delta}^H - B_t^H\right) \\
&= \frac{1}{2}\Big[(|s - t - \Delta|)^{2H} + (|s - t + \Delta|)^{2H} - 2|s - t|^{2H}\Big]. \qquad (3.9)
\end{aligned}$$

It immediately follows that the covariance structure of $B^{H,\Delta}$ is invariant under any shift $t \mapsto t + h$ with $h \geq 0$, meaning that for any lag $\Delta \geq 0$, any shift $h \geq 0$, and any sample points $t_1, \ldots, t_n$,

$$\left(B_{t_1+\Delta+h}^H - B_{t_1+h}^H, \ldots, B_{t_n+\Delta+h}^H - B_{t_n+h}^H\right) \sim \left(B_{t_1+\Delta}^H - B_{t_1}^H, \ldots, B_{t_n+\Delta}^H - B_{t_n}^H\right) \qquad (3.10)$$

Either from (3.10) or from (3.1) and (3.3) we deduce that for any lag $\Delta \geq 0$ and any sample point $t \geq 0$ we have $\mathrm{Var}\left(B_t^{H,\Delta}\right) = \Delta^{2H}$.

3.4 Autocovariance

One of the defining features of fBM with $H \neq \frac{1}{2}$, setting it apart from other widely studied processes (e.g. standard Brownian motion), is the fact that its increments have non-trivial covariance; and even more is true. In their seminal paper, B.B. Mandelbrot and J.W. Van Ness point out that:

> The basic feature of fBm's is that the "span of interdependence" between their increments can be said to be infinite. - [9, p. 422]

Recall that a stochastic process $(X_t)_{t\geq 0}$ is called **stationary** if $(X_{t+h})_{t\geq 0} \sim (X_t)_{t\geq 0}$ for any shift $h \in \mathbb{R}$.[3] While an fBM B^H is very much non-stationary (among other reasons because its variance is non-constant), its increment process $B^{H,\Delta}$ is - this was shown in (3.10). Recall that the **autocovariance function** ρ of a centred stationary stochastic process $(X_t)_{t\geq 0}$ is defined by

$$\rho(t-s) := \mathrm{Cov}(X_{t-s}, X_0) = \mathrm{Cov}(X_t, X_s), \quad 0 \leq s \leq t. \tag{3.11}$$

Requiring stationarity of X ensures that the second equality holds (by shifting with $-s$) and that ρ is well-defined. Note that due to the symmetry of $\mathrm{Cov}(\cdot,\cdot)$, we have $\rho(x) = \rho(|x|)$ for every $x \in \mathbb{R}$.

Definition 3.5. We say that a stationary stochastic process has long-range dependence if

$$\sum_{n=1}^{\infty} \rho(n) = \infty \quad .$$

Many stochastic processes, such as the increment process of Brownian motion (or any Levy process) or a Poisson process, exhibit no long range dependence. In fact, the autocovariance function of the increment process with lag $\Delta \geq 0$ of any stochastic process with stationary and *independent* increments is 0 unless $|s-t| < \Delta$. fBM, on the other hand, does not posses this property when $\frac{1}{2} < H < 1$, making it very attractive for modelling phenomena with correlation across long periods of time.

Lemma 3.6. *Let $B^H = (B^H_t)_{t\geq 0}$ be an fBM. Then $\left(B^H_{t+1} - B^H_t\right)_{t\geq 0}$ is stationary and*

$$\rho_{H,1}(x) = \frac{1}{2}\left(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\right) \sim H(2H-1)x^{2H-2}, \quad x \to \infty, \tag{3.12}$$

where $\rho_{H,1}$ is the autocovariance function of $(B^H_{t+1} - B^H_t)_{t\geq 0}$.

Proof. Stationarity was shown in (3.10). Substituting $x = s-t$ and $\Delta = 1$ in ((3.9)) gives the first equality. For the asymptotic, let $\varepsilon := \frac{1}{x}$. Then

$$\rho_{H,1}(x) = \frac{1}{2}x^{2H}\left[|1+\varepsilon|^{2H} + |1-\varepsilon|^{2H} - 2\right] \tag{3.13}$$

Taylor expanding $(1\pm\varepsilon)^{2H}$ around $\varepsilon = 0$ as

$$(1\pm\varepsilon)^{2H} = 1 \pm 2H\varepsilon + H(2H-1)\varepsilon^2 + o(\varepsilon^2)$$

and plugging it into (3.13) gives

$$\begin{aligned}\rho_{H,1}(x) &= \frac{1}{2}x^{2H}\left[1 + 2H\varepsilon + H(2H-1)\varepsilon^2 + o(\varepsilon^2) + 1 - 2H\varepsilon + H(2H-1)\varepsilon^2 + o(\varepsilon^2) - 2\right] \\ &= \frac{1}{2}x^{2H}\left[2H(2H-1)\varepsilon^2 + o(\varepsilon^2)\right], \quad x \gg 0 \quad .\end{aligned}$$

[3] For $h < 0$ the condition is understood to hold whenever it makes sense, i.e. when the shifted index remains in the indexing set as e.g. in formula (3.11).

Thus for $x \to \infty$ we have

$$\frac{\frac{1}{2}x^{2H}\left[2H(2H-1)\varepsilon^2 + o(\varepsilon^2)\right]}{x^{2H-2}H(2H-1)} = 1 + \underbrace{\frac{\frac{1}{2}x^{2H}o(\varepsilon^2)}{x^{2H}H(2H-1)\varepsilon^2}}_{\to 0} \to 1,$$

which proves the claim. □

Corollary 3.7. *(Long range dependence) Let $(B^H)_{t\geq 0}$ be an fBM. Then $(B^{H,\Delta})_{t\geq 0}$ exhibits long range dependence if and only if $\frac{1}{2} < H < 1$.*

Proof.

- If $H = \frac{1}{2}$, then $(B_t^H)_{t\geq 0}$ is standard Brownian motion and $\rho_{\frac{1}{2},1}$ is eventually 0.
- If $0 < H < \frac{1}{2}$, then $2H - 2 < -1$, then the tail of the function in (3.12) is summable.
- If $\frac{1}{2} < H < 1$, then $2H - 2 > -1$, then the tail is not summable.

□

More can be said about the general dependence of increments.

Proposition 3.8. *(Dependence of increments) Let $B^H = (B_t^H)_{t\geq 0}$ be an fBM and $0 \leq s_1 < t_1 < s_2 < t_2$. Then*

$$\operatorname{Cov}\left(B_{t_1}^H - B_{s_1}^H, B_{t_2}^H - B_{s_2}^H\right) \begin{cases} < 0, & \textit{if } \ 0 < H < \frac{1}{2} \quad \textit{"mean reverting"} \\ > 0, & \textit{if } \ \frac{1}{2} < H < 1 \quad \textit{"trending"} \end{cases}$$

Proof. If $0 < H < \frac{1}{2}$, then $x \mapsto x^{2H}$ is strictly concave, and thus

$$\begin{aligned} 0 &= \frac{1}{2}\Big[(t_2 - s_1) + (s_2 - t_1) - (t_2 - t_1) - (s_2 - s_1)\Big]^{2H} \\ &> \frac{1}{2}\Big[(t_2 - s_1)^{2H} + (s_2 - t_1)^{2H} - (t_2 - t_1)^{2H} - (s_2 - s_1)^{2H}\Big] \ . \end{aligned}$$

The case of $\frac{1}{2} < H < 1$ follows similarly, replacing concavity with convexity. □

At least for the increment process and in the asymptotic region this can also be seen from (3.12). Compare the above result to figures 1 to 5 in section 7.

4 Properties of Sample Paths

4.1 Continuity

While the continuity of fBM was postulated in its definition, we can say more about the regularity of its sample paths. The right tool to analyse the regularity of stochastic processes tends to be local Hölder-continuity or (almost equivalently) local p-variation. The central theorem in this direction is the Continuity Theorem of Kolmogorov–Centsov.

Theorem 4.1 (Kolmogorov–Centsov)**.** *Let $X = (X_t)_{0\le t\le T}$ be a stochastic process. If there exist constants $\alpha, \beta, C > 0$ s.t.*

$$\mathbb{E}\left[|X_t - X_s|^\alpha\right] \le C|t-s|^{1+\beta}, \quad 0 \le s, t \le T,$$

then there exists a continuous modification of X which is locally γ-Hölder-continuous for every $\gamma \in \left(0, \frac{\beta}{\alpha}\right)$. Since Hölder-continuity implies continuity, by Proposition 2.2, there exists only one such modification (up to indistinguishability).

Proof. See [8, Thm. 2.2.8]. □

Theorem 4.2. *Let $B^H = (B_t^H)_{t\ge 0}$ be an fBM. Then there exists a continuous modification of B^H which is locally γ-Hölder for $\gamma \in (0, H)$.*

Proof. Let $0 \le s \le t$ be arbitrary. Then because B^H is centred

$$\mathbb{E}\left[\left|B_t^H - B_s^H\right|^2\right] = \mathrm{Var}\left[B_t^H - B_s^H\right] = |t-s|^{2H},$$

and since B^H is Gaussian we have

$$B_s^H - B_t^H \sim B_1^H \sqrt{\mathbb{E}\left[|B_t^H - B_s^H|^2\right]},$$

and thus for any $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\left[\left|B_t^H - B_s^H\right|^n\right] &= \left(\sqrt{\mathbb{E}\left[|B_t^H - B_s^H|^2\right]}\right)^n \mathbb{E}\left[\left|B_1^H\right|^n\right] \\ &\le \left(|t-s|^{2H}\right)^{\frac{n}{2}} \mathbb{E}\left[\left|B_1^H\right|^n\right] \\ &= |t-s|^{Hn}\mathbb{E}\left[\left|B_1^H\right|^n\right].\end{aligned}$$

Thus, by Theorem 4.1, for each $n \in \mathbb{N}$ there exists a continuous modification $\widetilde{B^H}^{(n)}$ of B^H which is locally γ-Hölder continuous with $\gamma \in (0, \frac{Hn-1}{n})$. Since continuous modifications are unique up to indistinguishability by Proposition 2.2, there exists a single continuous modification which is locally γ-Hölder continuous for every $\gamma \in (0, H)$. □

Corollary 4.3. *Let $B^H = (B_t^H)_{t\ge 0}$ be an fBM. Then B^H does not have locally γ-Hölder sample paths for $\gamma \ge H$ on any set $A \subseteq \Omega$ with $\mathbb{P}(A) > 0$.*

Proof. If B^H were locally γ-Hölder for some $\gamma \ge H$ on a set A of positive measure, then in particular this would be true for $H = \gamma$. Thus on this set, the sample paths of B^H had finite $\frac{1}{H}$-variation: Since for any $T > 0$ and any finite partition $\mathcal{P}$ of $[0, T]$ we have

$$\sum_{t_i\in\mathcal{P}} \left|B_{t_i}^H(\omega) - B_{t_{i-1}}^H(\omega)\right|^{\frac{1}{H}} \le \sum_{t_i\in\mathcal{P}} \left\|B^H(\omega)\right\|_{H;[0,T]}^{\frac{1}{H}} |t_i - t_{i-1}| \le \left\|B^H(\omega)\right\|_{H;[0,T]}^{\frac{1}{H}} T, \quad \forall\omega \in A,$$

where $\|\cdot\|_{H;[0,T]}$ denotes the H-Hölder norm on $[0, T]$. Taking the supremum over all finite partitions $\mathcal{P}$ gives the result. However, [14] shows that the $\frac{1}{H}$-variation of fBM is infinite $\mathbb{P}$-a.s. which poses a contradiction. □

Corollary 4.4. *Let $B^H = \left(B_t^H\right)_{t\geq 0}$ be an fBM. Then B^H does not have differentiable sample paths on any set $A \subseteq \Omega$ with $\mathbb{P}(A) > 0$.*

Proof. If B^H were differentiable on a set A with $\mathbb{P}(A) > 0$, then on that set the sample paths would be of finite 1-variation. However, by Corollary 4.6 there exists a sequence of finite partitions along which the 1-variation diverges in L^2. Thus there exists a subsequence along which the 1-variation diverges $\mathbb{P}$-a.s. Since the 1-variation is greater than any 1-variation along a sequence of finite partitions, the 1-variation cannot be finite. □

4.2 Variation

Theorem 4.5. *([11, Thm. 2.1]) Let $f \in L^2(\mathbb{R}, \gamma_1)$, and let $B^H = (B_t^H)_{t\geq 0}$ be an fBM. Then*

$$\frac{1}{n}\sum_{k=1}^{n} f\left(B_k^H - B_{k-1}^H\right) \to \mathbb{E}\left[f\left(B_1^H\right)\right], \quad n \to \infty \quad \text{in } L^2(\Omega, \mathbb{P}). \tag{4.1}$$

Proof. "$H = \frac{1}{2}$": If $H = \frac{1}{2}$, then the increments $\left(B_k^H - B_{k-1}^H\right)_{k\in\mathbb{N}}$ are independent and $\mathcal{N}(0,1)$-distributed. Thus the sequence $\left(f\left(B_k^H - B_{k-1}^H\right)\right)_{k\in\mathbb{N}}$ is iid and square-integrable, and thus by the law large numbers (4.1) follows.

"$H \neq \frac{1}{2}$": Since $f \in L^2(\mathbb{R}, \gamma_1)$ we may expand f into the orthonormal basis given by the Hermite polynomials as

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k(f)}{\sqrt{k!}} H_k(x), \quad x \in \mathbb{R}.$$

The orthogonality of the Hermite polynomials gives

$$\mathbb{E}_{\mathbb{P}}\left[f(A)^2\right] = \mathbb{E}_{\gamma_1}\left[f^2\right] = \sum_{k=0}^{\infty} c_k(f)^2$$

and

$$\mathbb{E}\left[f(B_1^H)\right] = \left\langle f(B_1^H), 1\right\rangle_{L^2(\Omega,\mathbb{P})} = \langle f, 1\rangle_{L^2(\mathbb{R},\gamma_1)} = c_0(f). \tag{4.2}$$

Recall that we want to show that

$$\mathbb{E}\left[\left(\left[\frac{1}{n}\sum_{k=1}^{n} f\left(B_k^H - B_{k-1}^H\right)\right] - \mathbb{E}[f(B_1^H)]\right)^2\right] \to 0, \quad n \to \infty.$$

Firstly, by using (4.2), the fact that B_1^H is $\mathcal{N}(0,1)$-distributed, and the orthonormal basis expansion into Hermite polynomials, we deduce

$$\begin{aligned}\left[\frac{1}{n}\sum_{k=1}^{n} f\left(B_k^H - B_{k-1}^H\right)\right] - \mathbb{E}\left[f\left(B_1^H\right)\right] &= \frac{1}{n}\sum_{k=1}^{n}\Big[f\left(B_k^H - B_{k-1}^H\right) - \mathbb{E}\left[f\left(B_1^H\right)\right]\Big] \\ &= \frac{1}{n}\sum_{l=1}^{\infty}\frac{c_l(f)}{\sqrt{l!}}\sum_{k=1}^{n} H_l\left(B_k^H - B_{k-1}^H\right).\end{aligned}$$

Thus, using again the orthogonality of the Hermite polynomials,

$$\begin{aligned}&\mathbb{E}\left[\left(\left[\frac{1}{n}\sum_{k=1}^{n} f\left(B_k^H - B_{k-1}^H\right)\right] - \mathbb{E}\left[f\left(B_1^H\right)\right]\right)^2\right] = \\ &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{l=1}^{\infty}\frac{c_l(f)}{\sqrt{l!}}\sum_{k=1}^{n} H_l\left(B_k^H - B_{k-1}^H\right)\right)^2\right] \\ &= \frac{1}{n^2}\sum_{l,m=1}^{\infty}\frac{c_l(f)c_m(f)}{\sqrt{l!m!}}\mathbb{E}\left[\left(\sum_{k=1}^{n} H_l\left(B_k^H - B_{k-1}^H\right)\right)\left(\sum_{h=1}^{n} H_m\left(B_h^H - B_{h-1}^H\right)\right)\right] \\ &= \frac{1}{n^2}\sum_{l,m=1}^{\infty}\frac{c_l(f)c_m(f)}{\sqrt{l!m!}}\sum_{k,h=1}^{n}\mathbb{E}\left[H_l\left(B_k^H - B_{k-1}^H\right)H_m\left(B_h^H - B_{h-1}^H\right)\right]\end{aligned}$$

By Lemma 2.5 and the fact that for any $k \in \mathbb{N}$ the random variable $B_k^H - B_{k-1}^H \sim B_1^H$ is $\mathcal{N}(0,1)$-distributed[4] we obtain

$$\frac{1}{n^2}\sum_{l=1}^{\infty} c_l(f)^2 \sum_{k,h=1}^{n}\mathbb{E}\left[\left(B_k^H - B_{k-1}^H\right)\left(B_h^H - B_{h-1}^H\right)\right]^l.$$

Since $(B_t^H - B_{t-1}^H)_{t\geq 0}$ is stationary by Proposition 3.2 (4), this yields

$$\frac{1}{n^2}\sum_{l=1}^{\infty} c_l(f)^2 \sum_{k,h=1}^{n}\rho_{H,1}(|k-h|)^l,$$

where $\rho_{H,1}$ is the autocovariance function of the increment process of fBM with increment length 1. Via the Cauchy–Schwarz inequality and stationarity of increments we have

$$|\rho_{H,1}(|k-h|)| = \mathbb{E}\left[\left(B_k^H - B_{k-1}^H\right)\left(B_h^H - B_{h-1}^H\right)\right] \leq \underbrace{\sqrt{\mathrm{Var}(B_k^H - B_{k-1}^H)}\sqrt{\mathrm{Var}(B_h^H - B_{h-1}^H)}}_{\leq\sqrt{1}\cdot\sqrt{1}} \leq 1$$

and thus

$$|\rho_{H,1}(|k-h|)|^l \leq |\rho_{H,1}(|k-h|)|, \quad l \in \mathbb{N}.$$

This finally leads to

[4] Here we use stationarity of increments.

$$\mathbb{E}\left[\left(\left[\frac{1}{n}\sum_{k=1}^{n} f\left(B_k^H - B_{k-1}^H\right)\right] - \mathbb{E}[f(B_1^H)]\right)^2\right] \leq \frac{1}{n^2}\underbrace{\sum_{l=1}^{\infty} c_l(f)^2}_{=\mathrm{Var}[f(B_1^H)]} \sum_{k,h=1}^{n} |\rho(|k-h|)|$$

$$\leq \mathrm{Var}\left[f\left(B_1^H\right)\right]\frac{1}{n^2}\sum_{h=1}^{n}\underbrace{\sum_{k=1-h}^{n-h}|\rho(|k|)|}_{\leq 2\sum_{k=0}^{n-1}|\rho(|k|)|} \leq 2\,\mathrm{Var}\left[f\left(B_1^H\right)\right]\frac{1}{n}\sum_{k=0}^{n-1}|\rho(|k|)|.$$

Thus the problem is reduced to studying the asymptotic behaviour of $\frac{1}{n}\sum_{k=0}^{n-1}|\rho(|k|)|$. By Lemma 3.6 we have

$$|\rho(|k|)| \sim H(2H-1)k^{2H-2}, \quad k \to \infty \quad .$$

Thus if $H < \frac{1}{2}$, then

$$\sum_{k=0}^{n}|\rho(|k|) \sim \sum_{k=0}^{n} H(2H-1)k^{2H-2} \leq \sum_{k=0}^{\infty} H(2H-1)k^{2H-2} < \infty$$

and hence $\frac{1}{n}\sum_{k=0}^{n-1}|\rho(|k|)| \to 0$ as $n \to \infty$.

If $H > \frac{1}{2}$, then

$$\frac{1}{n}\sum_{k=0}^{n-1}|\rho(|k|) \sim H(2H-1)\frac{1}{n}\sum_{k=0}^{n-1}k^{2H-2} \sim H\frac{1}{n}n^{2H-1} \to 0, \quad n \to \infty.$$

To see the last asymptotic equivalence, note that

$$\sum_{k=0}^{n-1}k^{2H-2} \leq \int_0^n k^{2H-2}\mathrm{d}k = \frac{1}{2H-1}n^{2H-1} \quad .$$

□

This gives the computation of p-variation as a corollary:

Corollary 4.6. *Let $B^H = (B_t^H)_{t\geq 0}$ be an fBM and $1 \leq p < \infty$. Then for any $t \geq 0$*

$$\sum_{k=1}^{n}\left|B^H_{\frac{k}{n}\cdot t} - B^H_{\frac{k-1}{n}\cdot t}\right|^p \to \begin{cases} 0 & , \quad \frac{1}{H} < p \\ t\mathbb{E}[|B_1^H|^p] & , \quad \frac{1}{H} = p \\ +\infty & , \quad \frac{1}{H} > p \end{cases} \quad \text{in } L^2(\Omega, \mathbb{P}).$$

In other words, the p-variation along the sequence of finite partitions $\{\frac{k}{n} : 0 \leq k \leq n\}_{n\in\mathbb{N}}$ is finite if and only if $\frac{1}{H} \leq p$ and 0 if and only if $\frac{1}{H} < p$. Note that this does *not* mean that

$$\sup_{\mathcal{P}}\sum_{t_i\in\mathcal{P}}\left|B^H_{t_i} - B^H_{t_{i-1}}\right|^p$$

is finite, where $\mathcal{P}$ ranges over all finite partitions of $[0,t]$ - see [14].

Proof of Cor. 4.6. Since fBM is self-similar, $\left(\left(\frac{t}{n}\right)^{-H} B^H_{\frac{k}{n}\cdot t}\right)_{k\geq 0}$ is also an fBM. So by Theorem 4.5 and choosing $f : x \mapsto |x|^p$ we get

$$\frac{1}{n}\sum_{k=1}^{n}\left|\left(\frac{t}{n}\right)^{-H} B^H_{\frac{k}{n}\cdot t} - \left(\frac{t}{n}\right)^{-H} B^H_{\frac{k-1}{n}\cdot t}\right|^p \to \mathbb{E}\left[\left|B^H_1\right|^p\right] \quad \text{in } L^2(\Omega, \mathbb{P}).$$

Thus also

$$t^{-pH} n^{pH-1}\left(\sum_{k=1}^{n}\left|B^H_{\frac{k}{n}\cdot t} - B^H_{\frac{k-1}{n}\cdot t}\right|^p\right) \to \mathbb{E}\left[\left|B^H_1\right|^p\right] \quad \text{in } L^2(\Omega, \mathbb{P}).$$

The conclusion now follows since

$$n^{pH-1} \to \begin{cases} 0 & , \quad \frac{1}{H} > p \\ 1 & , \quad \frac{1}{H} = p \\ +\infty & , \quad \frac{1}{H} < p \end{cases}$$

and thus

$$\sum_{k=1}^{n}\left|B^H_{\frac{k}{n}\cdot t} - B^H_{\frac{k-1}{n}\cdot t}\right|^p \to \begin{cases} 0 & , \quad \frac{1}{H} < p \\ t\mathbb{E}[|B^H_1|^p] & , \quad \frac{1}{H} = p\,. \\ +\infty & , \quad \frac{1}{H} > p \end{cases}$$

□

5 Probabilistic Properties

We now turn to some of the probabilistic properties of fBM.

5.1 Law of Large Numbers

Recall that for standard Brownian motion

$$\frac{B_n}{n} \to 0 \quad \mathbb{P}-\text{a.s.}, \quad n \to \infty,$$

The classical law of large numbers is not applicable to fBM since its increments are not independent. However, as an immediate corollary of Theorem 4.5 we get a law of large numbers for fractional Brownian motion as follows.

Corollary 5.1. *(Law of Large Numbers) Let $B^H = (B^H_t)_{t\geq 0}$ be an fBM. Then*

$$\frac{B^H_n}{n} \to 0, \quad \text{in } L^2(\Omega, \mathbb{P}).$$

Proof. Set $f : x \mapsto x$ in Theorem 4.5. □

5.2 Law of Iterated Logarithm

Recall that for standard Brownian motion the law of the iterated logarithm gives precise tail bounds:

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \ln\ln t}} = 1 \ \ \mathbb{P} - a.s.$$

See for example [8, Thm. 2.9.23]. A similar result holds for fractional Brownian motion:

Theorem 5.2. *(Law of the Iterated Logarithm) Let $B^H = (B^H_t)_{t\geq 0}$ be an fBM. Then*

$$\limsup_{t\to\infty} \frac{B^H_t}{\sqrt{2t^{2H} \ln\ln t}} = 1 \ \ \mathbb{P} - a.s.$$

Proof. See [13, Thm. 1.1 (C)] with $Y = B^H$ and hence $Q(t) = \frac{1}{2}t^{2H}$. Then choosing $v(t) = t^{2H}$, $s_0 = 0$, and $\beta_1 = \beta_2 = \beta_3 = 2H$ satisfies the conditions gives the result. □

5.3 Semimartingale Property

Recall that a stochastic process (X_t) is called a (continuous) **semi-martingale** if

$$X_t = M_t + A_t, \quad 0 \leq t,$$

where (M_t) is a (continuous) local martingale and (A_t) is a cadlag adapted process which is locally of bounded variation. These processes are of course very important as the set of continuous local martingales is the largest class of integrators for which there exists an Ito theory of integration. However, unfortunately, fBM does not lie in this class for any $H \neq \frac{1}{2}$.

Theorem 5.3. *Every continuous semi-martingale has finite 2-variation along a sequence of partitions and strictly positive 2-variation along a sequence unless it has finite total variation. The convergence is to be taken in probability.*

Proof. Recall that the quadratic variation of a semi-martingale is equal to the quadratic variation of the local martingale part. Then the result follows from [8, Thm 5.8]. □

Theorem 5.4. *Let $B^H = (B^H_t)_{t\geq 0}$ be an fBM. Then B^H is a semi-martingale if and only if $H = \frac{1}{2}$, in which case it is a martingale.*

Proof. If $H = \frac{1}{2}$, B^H is standard Brownian motion and is thus a continuous martingale. Let $H \neq \frac{1}{2}$ and assume B^H was a martingale. Then Corollary 4.6 and Theorem 5.3 would be in contradiction to one another. □

5.4 Markov Property

Another aspect that sets fBM apart from standard Brownian motion is the following.

Theorem 5.5. *([11, Thm. 2.3]) Let $B^H = (B^H_t)_{t\geq 0}$ be an fBM. Then B^H is Markov if and only if $H = \frac{1}{2}$.*

Proof. "$\Rightarrow$": Let $H \neq \frac{1}{2}$ and assume B^H were Markov. Then since the process is Gaussian, for any $0 < s < t < u$ we have

$$\operatorname{Cov}(B_s^H, B_u^H) \operatorname{Cov}(B_t^H, B_t^H) = \operatorname{Cov}(B_s^H, B_t^H) \operatorname{Cov}(B_t^H, B_u^H), \tag{5.1}$$

(see [1, p. 168, Thm. 8.4]). Choose $u = 1$ and define the function

$$\kappa(s) := \operatorname{Cov}\left(B_s^H, B_1^H\right) = \frac{1}{2}\left(1 + s^{2H} - (1-s)^{2H}\right) > 0, \quad 0 \leq s \leq 1. \tag{5.2}$$

Note that for any $0 \leq s < t \leq 1$ we have $0 \leq \frac{s}{t} \leq 1$ and

$$\operatorname{Cov}\left(B_t^H, B_t^H\right) \kappa\left(\frac{s}{t}\right) = t^{2H} \frac{1}{2}\left[1 + \left(\frac{s}{t}\right)^{2H} - \left(1 - \left(\frac{s}{t}\right)\right)^{2H}\right] = \operatorname{Cov}\left(B_s^H, B_t^H\right),$$

and thus via equation (5.1)

$$\kappa\left(\frac{s}{t}\right) = \frac{\kappa(s)}{\kappa(t)}.$$

Define

$$\psi(s) = \ln(\kappa(e^{-s})), \quad 0 \leq s,$$

and observe that

$$\psi(0) = 0, \quad \lim_{s \to \infty} \psi(s) = -\infty, \quad \text{and} \quad \psi(s+t) = \psi(s) + \psi(t), \quad s, t \geq 0. \tag{5.3}$$

Differentiating the latter w.r.t. s yields $\psi'(s+t) = \psi'(t)$ for every $s \geq 0$, implying that ψ' is constant and thus ψ is linear on $[0, \infty)$. Via the second observation in line (5.3) this means that there exists an $\alpha > 0$ s.t. $\psi(s) = -\alpha s, s \in \mathbb{R}$, or equivalently

$$\kappa(s) = s^{\alpha}, \quad 0 < s \leq 1. \tag{5.4}$$

Differentiating the term in equation (5.2) twice gives

$$|\kappa''(s)| = \left| H(2H-1)\left(\underbrace{s^{2H-1}}_{\to 1} - \underbrace{(1-s)^{2H-2}}_{\to \infty}\right)\right| \to \infty, \quad s \to 1,$$

since

$$H \neq \frac{1}{2} \Rightarrow H(2H-1) \neq 0 \quad \text{and} \quad H < 1 \Rightarrow 2H - 2 < 0.$$

But this is a contradiction to formula (5.4), since it implies

$$|\kappa''(s)| \to |\kappa(1)| = \alpha|\alpha - 1| < \infty, \quad s \to 1.$$

"$\Leftarrow$": Assume $H = \frac{1}{2}$. Then B^H is Brownian motion and thus Markov. $\square$

6 Further Topics

6.1 Integral w.r.t. fractional Brownian Motion

In order to solve stochastic differential equations driven by fractional Brownian motion

$$\mathrm{d}X_t = b(t, X_t)dt + \sigma(t, X_t)\mathrm{d}B_t^H$$

one needs a notion of integral w.r.t. B^H. If $H = \frac{1}{2}$ such an integral is provided by Ito's theory of stochastic integration. However, as shown in section 5.3, whenever $H \neq \frac{1}{2}$, B^H is not a semi-martingale and the theory is not available. The right replacement depends on whether we are in the rough case ($0 < H < \frac{1}{2}$) or in the regular case ($\frac{1}{2} < H < 1$).

In the latter case, Young's theory of integration is a natural choice (for sufficiently regular integrands), while in the former case, more refined methods, like the theory of rough paths is needed. Yet another alternative is to use the fact that fBM is a Gaussian process and utilize the Malliavin calculus and the Skohorod integral.

We will consider none of the above here and instead refer to [6], [2], [12], [5].

6.2 Fractional Calculus

The expression in formula (3.5) is (essentially) a fractional integral. This is no coincidence. There is a rich theory involving fBM and fractional calculus, relating fBMs of different Hurst parameter to one another in a similar way to how formula (3.5) relates fBM to standard Brownian motion - see [10]. Interestingly though, fBM does not get its name from its definition via a fractional integral but from the "fractional nature" of its spectral density λ^{1-2H} when $H \neq \frac{1}{2}$ - see [9, p. 422].

6.3 Cameron-Martin space

The following is a reformulation of [4, Thm. 41, 44]. As a continuous Gaussian process, fBM on $[0, 1]$ has a Cameron-Martin space. It can be given as the image of $L^2([0, 1], \lambda)$ under the map

$$h \mapsto \int_0^1 K_H(t, s)h(s)\mathrm{d}s, \quad h \in L^2([0, 1], \lambda),$$

where

$$K_H(t, s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\right),$$

F is the Gaussian hypergeometric function, and Γ is the usual gamma function. See also [5].

7 Simulation of fBM and Sample Paths

The following is (up to small modifications) taken from [15, Sec. 6]. Many other methods of simulation can be found in [3]. The simulation was done in MATLAB. We will give it here without further explanation.

```
H = 0.5 % Hurst parameter 0 < H < 1
q = 10  % dyadic level of partition
n = 3   % number of realizations
T = 1   % time horizon

N = 2^q+1                 % total number of time sample points
time = (T/(N-1))*(0:N-1) % normalized to time interval [0,T]

lambda = Lambda(H,N)
fGnsamples = [zeros(n,1) FGN(lambda ,n)]
simulation_data = fBM_sim(T,N,H,fGnsamples)
plot(time,simulation_data)

function res = Lambda(H,N)
M = 2*N - 2;
C = zeros(1,M);
G = 2*H;
fbc = @(n)((n+1).^G + abs(n -1).^G - 2*n.^G)/2;
C(1:N) = fbc(0:(N -1));
C(N+1:M) = fliplr(C(2:(N -1)));
res = real(fft(C)).^0.5;
end

function res = FGN(lambda ,NT)
if (~exist('NT','var'))
NT = 1;
end
M = size(lambda ,2);
a = bsxfun(@times ,ifft(randn(NT ,M),[],2),lambda );
res =   real(fft(a,[] ,2));
res = res(: ,1:(M/2));
end

function res = fBM_sim(T,N,H,fGnsamples)
        increments = (T/N)^H * fGnsamples;
        res = cumsum(increments,2);
end
```

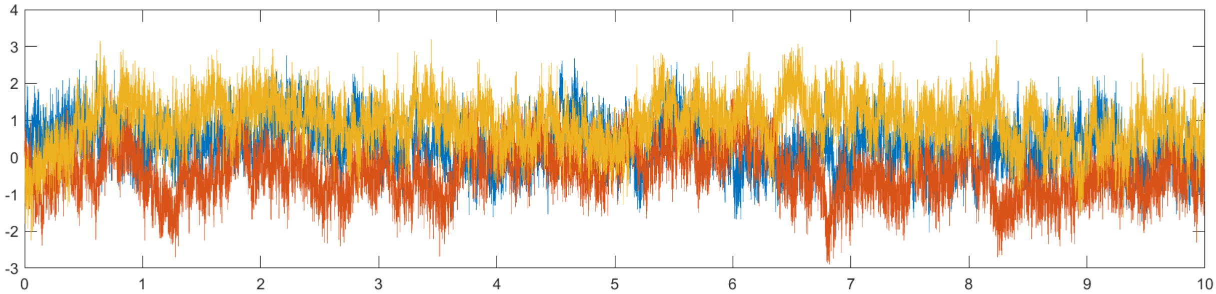

Figure 1: $H = 0.1$

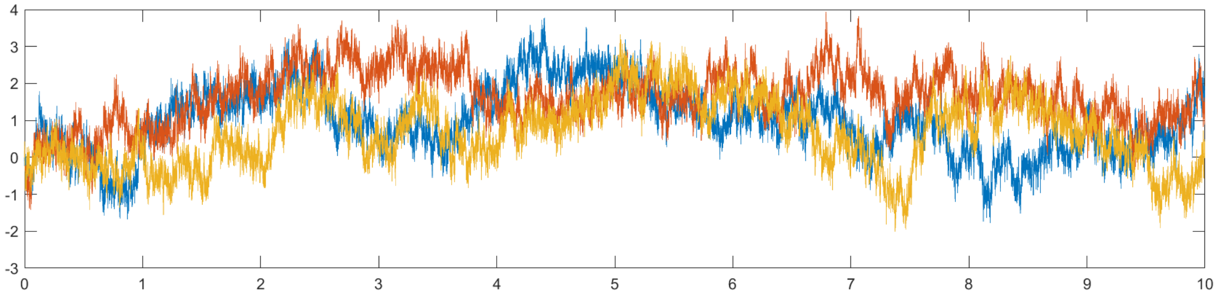

Figure 2: $H = 0.2$

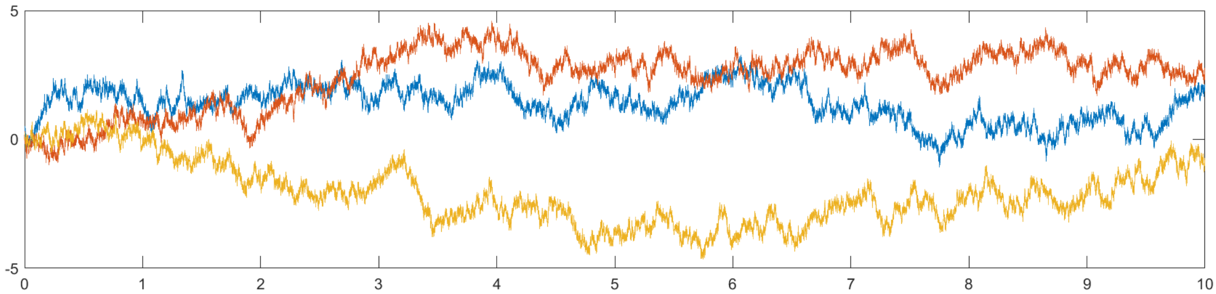

Figure 3: $H = 0.3$

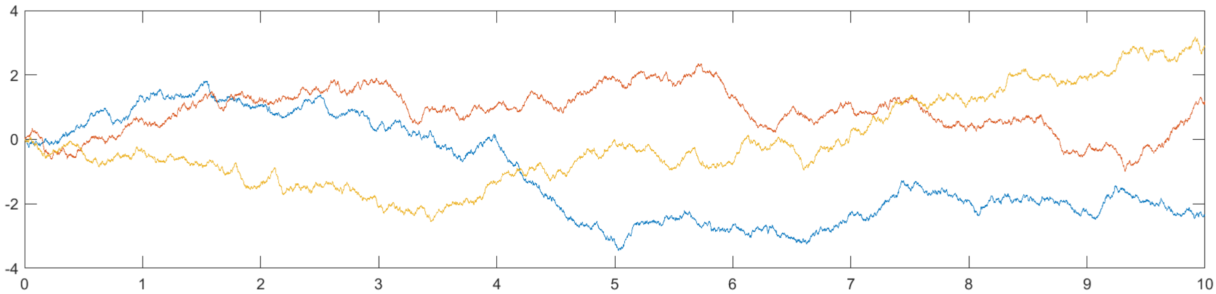

Figure 4: $H = 0.6$

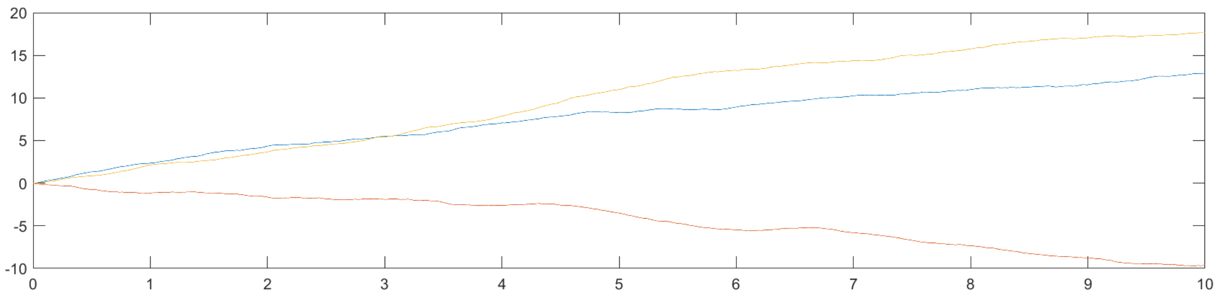

Figure 5: $H = 0.9$

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